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Maximal function on Musielak–Orlicz spaces and generalized Lebesgue spaces

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Abstract

We consider the Hardy–Littlewood maximal operator M on Musielak–Orlicz Spaces $L^{\varphi}(\mathbb{R}^d)$. We give a necessary condition for the continuity of M on $L^{\varphi}(\mathbb{R}^d)$ which generalizes the concept of Muckenhoupt classes. In the special case of generalized Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^d)$ we show that this condition is also sufficient. Moreover, we show that the condition is “left-open” in the sense that not only M but also M_q is continuous for some $q > 1$, where $M_q f = (M(|f|^q))^{1/q}$.

Résumé

On considère l’opérateur maximal Hardy–Littlewood sur les espaces Musielak–Orlicz $L^{\varphi}(\mathbb{R}^d)$. Une condition nécessaire est donnée pour la continuité de M sur $L^{\varphi}(\mathbb{R}^d)$, qui généralise la conception des classes Muckenhaupt. Pour le cas spécial des espaces Lebesgues généralisés $L^{p(\cdot)}(\mathbb{R}^d)$ on justifie que cette condition est aussi suffisante. En plus, on prouve que la condition est “left-open” au sens que non seulement M mais aussi M_q est continu pour certes $q > 1$, où $M_q f = (M(|f|^q))^{1/q}$.

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1. Introduction

In recent years, the generalized Lebesgue spaces $L^{p(\cdot)}$ (also known as Lebesgue spaces with variable exponent, or $L^{p(x)}$) and the corresponding generalized Sobolev spaces $W^{1,p(\cdot)}$ have attracted more and more attention. The growing interest in this field is strongly stimulated by the treatment of recent problems in elasticity, fluid dynamics, calculus of variations, and differential equations with $p(x)$-growth. For example, in Růžička [29] one can find a model of electrorheological fluids, where the essential part of the dissipative energy is given by $\int |\mathbf{Df}(x)|^{p(x)}\,\mathrm{d}x$. Here $\mathbf{Df}$ denotes the symmetric part of the gradient $\nabla\mathbf{f}$. The same type of energy also appears in a model proposed by Zhikov [34] for another type of fluid, where the stress tensor depends on a distribution of temperature. This energy also appears in the investigations of variational integrals with non-standard growth, see e.g. Zhikov [32], Marcellini [21], Acerbi, Mingione [1]. The spaces $L^{p(\cdot)}$ provide the right setting for theses energies, i.e. $f \in L^{p(\cdot)}$ if and only if $\int |f(x)|^{p(x)}\,\mathrm{d}x < \infty$. We refer to Hudzik [16], Kováčik, Rákosník [19], Samko [30], Edmunds, Lang, Nekvinda [11], Růžička [29], Edmunds, Rákosník [12], Fan, Shen, Zhao [14], Diening [5,6] for basic properties of the spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$ such as reflexivity, denseness of smooth functions, and Sobolev type embeddings. The spaces $L^{p(\cdot)}$ (see Orlicz [27]) are special cases of the generalized Orlicz spaces L^{φ} originated by Nakano [24] and developed by Musielak and Orlicz [22,23], where $f \in L^{\varphi}$ if and only if $\int \varphi(x, |f(x)|)\,\mathrm{d}x < \infty$ for a suitable φ. We are strongly convinced that these more general spaces will become increasingly important in the modeling of modern materials.

Unfortunately the spaces L^{φ} and $L^{p(\cdot)}$ have some undesired properties. The translation operator for example is in general not continuous on L^{φ}. Let us be more precise in the case of $L^{p(\cdot)}$: For every $L^{p(\cdot)}$ with p non-constant exists a function $f \in L^{p(\cdot)}$ and a translation τ_h, such that $\tau_h f \notin L^{p(\cdot)}$ (see [6,19]). As a consequence the convolution of f with a function $g \in L^1$ is in general not continuous, i.e. in general $\|f * g\|_{p(\cdot)} \not\leqslant C\|g\|_1 \|f\|_{p(\cdot)}$ (failure of Young's inequality). Since these two very important tools – translation and convolution – fail on $L^{p(\cdot)}$ many of the standard results for classical Lebesgue and Sobolev spaces do not hold for the generalized Lebesgue and Sobolev spaces. Other results hold true, but need different, more subtile proofs. It is important to note that even the basic properties mentioned above, e.g. density of smooth functions [6,30] and Sobolev embeddings [7,13,14], are by no means trivial.

Despite the failure of translation and Young's inequality, it is surprising to find that under some restrictions on p it is still possible to mollify: If p satisfies the uniform local continuity condition

$$\bigl|p(x) - p(y)\bigr| \leqslant \frac{C}{|\ln|x-y||}, \tag{1.1}$$

it is still possible to mollify with $\varphi \in C_0^{\infty}(\mathbb{R}^d)$ functions [30,33]. One can reduce this property to the continuity of the Hardy–Littlewood maximal operator M (see [6]). If p satisfies the uniform local continuity condition (1.1) and additionally the decay condition

$$\bigl|p(x) - p_{\infty}\bigr| \leqslant \frac{C}{|\ln(\mathrm{e} + |x|)|} \tag{1.2}$$

for some $p_\infty \in (1,\infty)$, then M is continuous on $L^{p(\cdot)}(\mathbb{R}^d)$ [3,6,25]. From the continuity of M on $L^{p(\cdot)}(\mathbb{R}^d)$ if follows that $f * \varphi_\varepsilon \to f$ in $L^{p(\cdot)}(\mathbb{R}^d)$, where $\varphi_\varepsilon(x) := \varepsilon^{-d}\varphi(x/\varepsilon)$, for a large class of mollifiers including $C_0^\infty(\mathbb{R}^d)$. This immediately implies denseness of $C^\infty(\overline{\Omega})$ in $W^{1,p(\cdot)}(\Omega)$ for domains Ω with Lipschitz boundary. These are not the only results for $L^{p(\cdot)}(\mathbb{R}^d)$ which are based on the maximal operator. In [8] several results have been shown based on the sole condition that M is continuous on the spaces $L^{p(\cdot)}(\mathbb{R}^d)$, $L^{p'(\cdot)}(\mathbb{R}^d)$, $L^{p(\cdot)/s}(\mathbb{R}^d)$, and $L^{(p(\cdot)/r)'}(\mathbb{R}^d)$ for some $0 < r < 1 < s$: First, the fundamental estimate of Fefferman–Stein is generalized to $L^{p(\cdot)}(\mathbb{R}^d)$, i.e. $c\|f^\sharp\|_{p(\cdot)} \leqslant \|f\|_{p(\cdot)} \leqslant C\|f^\sharp\|_{p(\cdot)}$ for all $f \in L^{p(\cdot)}(\mathbb{R}^d)$, where $f^\sharp(x) := \sup_{Q\ni x} \fint_Q |f - (f)_Q|\,\mathrm{d}x$, $(f)_Q := \fint f\,\mathrm{d}y$. Second, Korn's inequality $\|\nabla\|_{p(\cdot)} \leqslant C\|\mathbf{Du}\|_{p(\cdot)}$ is proved, where $\mathbf{Df}$ is the symmetric part of the gradient $\nabla\mathbf{f}$. Third, it is shown that the divergence equation $\operatorname{div}\mathbf{u} = f$ possesses a solution $\mathbf{u} \in W_0^{1,p(\cdot)}(\Omega)$ satisfying the estimate $\|\nabla\mathbf{u}\|_{p(\cdot)} \leqslant C\|f\|_{p(\cdot)}$. All these results are only based on the continuity of M on $L^{p(\cdot)}(\mathbb{R}^d)$, $L^{p'(\cdot)}(\mathbb{R}^d)$, $L^{p(\cdot)/s}(\mathbb{R}^d)$, and $L^{(p(\cdot)/r)'}(\mathbb{R}^d)$ for some $0 < r < 1 < s$. It is therefore of interest to study necessary and sufficient conditions on p such that M is continuous on $L^{p(\cdot)}(\mathbb{R}^d)$ or more general continuous on $L^\varphi(\mathbb{R}^d)$. This is the content of this article.

After some preliminaries in Section 2 we present in Section 3 a necessary condition for the continuity of M on $L^\varphi(\mathbb{R}^d)$ which generalizes the concept of Muckenhoupt classes. We refer to this condition as "φ is of class $\mathcal{A}$" (see Definition 3.1). In the context of classical weighted Lebesgue spaces this condition agrees with the classical Muckenhoupt condition. In Section 4 we provide an alternative characterization of class $\mathcal{A}$ for the special case $L^{p(\cdot)}(\mathbb{R}^d)$ which can be more easily verified. We will see in Section 5 that class $\mathcal{A}$ has a self improving property, which is similar to the reverse Hölder estimates for classical, weighted Lebesgue spaces. In Section 6 we introduce a (possibly slightly stronger) condition which is sufficient to ensure the continuity of M on $L^\varphi(\mathbb{R}^d)$. We will see that this condition even implies the continuity of M_q for some $q > 1$, were $M_q f = (M(|f|^q))^{1/q}$. The existence of such $q > 1$ is the analog of the left-openness of the Muckenhoupt classes. In Section 7 we will characterize both the necessary condition "class $\mathcal{A}$" and the sufficient condition in a pointwise sense, i.e. similar to the characterization of embeddings of Musielak–Orlicz spaces in [22]. We will also discuss in detail the possible difference between the necessary and the sufficient condition in the general case. Nevertheless in the special case $L^{p(\cdot)}(\mathbb{R}^d)$ we will see in Section 8 that both conditions are equivalent, especially "class $\mathcal{A}$" is necessary AND sufficient. We will use our results to weaken the assumptions on p for the continuity of Calderón–Zygmund operators $L^{p(\cdot)}(\mathbb{R}^d)$, for Korn's inequality, and estimates for the solution of the divergence equation.

Let us make a comment on the existing conditions for $L^{p(\cdot)}(\mathbb{R}^d)$. As mentioned above the uniform local continuity condition (1.1) together with the decay condition (1.2) is sufficient to ensure the continuity of M on $L^{p(\cdot)}(\mathbb{R}^d)$. It is shown in [28] that this condition is the tightest condition in terms of a uniform local continuity modulus: If $\rho(t)$ is a concave continuity modulus with $\rho(t)|\ln t| \to \infty$ for $t \to 0^+$, then there exist $p:\mathbb{R} \to (1,\infty)$ which is ρ-continuous and constant outside some large ball such that M is not continuous on $L^{p(\cdot)}(\mathbb{R})$. On the other hand it is shown in [3] that the decay condition (1.2) cannot be replaced by a tighter decay condition, i.e. if $\gamma : [0,\infty) \to [0,\infty)$ is a decreasing decay condition with $\rho(t)\ln(t) \to \infty$ if $t \to \infty$, then there exists $p:\mathbb{R} \to (1,\infty)$ which satisfies the decay $|p(x) - p_\infty| \leqslant \gamma(t)$ for t sufficiently large so that M is not continuous on $L^{p(\cdot)}(\mathbb{R})$. (Note that this result is solely based on the decay and not some local regularity of p.) Due to these facts the uniform local condition (1.1) together with the decay condition (1.2) is widely accepted among researches and sometimes misleadingly called "necessary". Nevertheless these conditions are not necessary in the sense that they can be deduced from the continuity of M on $L^{p(\cdot)}(\mathbb{R}^d)$. Indeed, A. Nekvinda [26] has recently found a continuous function $p : \mathbb{R} \to [2, 3]$ such that M is continuous on $L^{p(\cdot)}(\mathbb{R})$. This function fails, however, the local continuity condition (1.1). We even conjecture that there exists a function p which is not continuous and has no limit $\lim_{|x|\to\infty} p(x)$ for which M is still continuous on $L^{p(\cdot)}(\mathbb{R})$. In this paper we will present a condition which is necessary and sufficient for M to be continuous on $L^{p(\cdot)}(\mathbb{R}^d)$. Moreover, we will show that M is continuous on $L^{p(\cdot)}(\mathbb{R}^d)$ if and only if it is continuous on $L^{p'(\cdot)}(\mathbb{R}^d)$.

2. Preliminaries

Let $\mathbb{R}^{\geqslant 0} := \{t \in \mathbb{R}\colon\ t \geqslant 0\}$ and $\mathbb{R}^{>0} := \{t \in \mathbb{R}\colon\ t > 0\}$. By C (without an index) we denote a positive constant which may change from line to line. Let $\mathcal{F}^d$ denote the set of all Lebesgue real valued, measurable functions on $\mathbb{R}^d$. By $\mathcal{X}^d$ we denote the set of all open cubes in $\mathbb{R}^d$ and by $\mathcal{Y}^d$ we denote the set of all families of disjoint, open cubes in $\mathbb{R}^d$. By X^* we denote the dual of the Banach space X.

Definition 2.1. Let Ω either denote $\mathbb{R}^d$, N, or $\mathcal{X}^d$. A real function $\varphi : \Omega \times \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}$ will be said to be an N-function on Ω if it satisfies the following conditions:

(a) There exists $a : \Omega \times \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}$ with $a(\omega, 0) = 0$, $a(\omega, t) > 0$ for $t > 0$ such that $a(\omega, \cdot)$ is a right-continuous, non-decreasing function for every $\omega \in \Omega$. Moreover, for all $\omega \in \Omega$

$$\varphi(\omega, t) = \int_0^t a(\omega, u)\,\mathrm{d}u.$$

(b) If $\Omega = \mathbb{R}^d$ we additionally require that $\varphi(x, t)$ is Lebesgue-measurable in x for all $t > 0$.

We usually write $\varphi'(x, t)$ instead of $a(x, t)$. We say that φ satisfies the strong Δ_2-condition, if there exists $C_1 > 0$ such that for all $\omega \in \Omega$ and all $t \geqslant 0$ holds $\varphi(\omega, 2t) \leqslant C_1 \varphi(\omega, t)$.

If φ is an N-function on $\mathbb{R}^d$ which satisfies the strong Δ_2-condition, then

$$L^\varphi(\mathbb{R}^d) := \left\{ f \in \mathcal{F}^d\colon \int_{\mathbb{R}^d} \varphi\big(x, |f(x)|\big)\,\mathrm{d}x < \infty \right\}$$

equipped with the norm (the Luxemburg-norm)

$$\|f\|_\varphi := \inf\left\{\lambda > 0\colon \int\limits_{\mathbb{R}^d} \varphi\left(x, \frac{|f(x)|}{\lambda}\right) \mathrm{d}x \leqslant 1\right\}$$

defines a Banach space (even more a Banach function space). The spaces $L^\varphi(\mathbb{R}^d)$ are special Musielak–Orlicz spaces (see [20,22]). Let $\varphi, \psi : \Omega \times \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}$. We say that $\varphi(\omega, t) \sim \psi(\omega, t)$ or $\varphi \sim \psi$ if there exist constants $c_2, C_2 > 0$ independent of $\omega \in \Omega$ and $t \geqslant 0$ such that

$$c_2\varphi(\omega, t) \leqslant \psi(\omega, t) \leqslant C_2\varphi(\omega, t)$$

for all $\omega \in \Omega$ and $t \geqslant 0$. Therefore the strong Δ_2-condition can we written as $\varphi(\omega, t) \sim \varphi(\omega, 2t)$. The following results are standard in the context of N-function (see [22]). By $(\varphi')^{-1} : \Omega \times \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}$ we denote the function

$$(\varphi')^{-1}(\omega, t) := \sup\bigl\{u \in \mathbb{R}^{\geqslant 0}\colon \varphi'(\omega, u) \leqslant t\bigr\}.$$

Then $\varphi^* : \Omega \times \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}$ with

$$\varphi^*(\omega, t) := \int\limits_0^t (\varphi')^{-1}(\omega, u)\,\mathrm{d}u$$

is again an N-function on Ω. It is the complementary function of φ. Note that $(\varphi^*)^* = \varphi$. For all $\omega \in \Omega$ and $t, u \geqslant 0$ holds

$$tu \leqslant \varphi(\omega, t) + \varphi^*(\omega, u). \tag{2.1}$$

Let $\varphi^{-1} : \Omega \times \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}$ (resp., $(\varphi^*)^{-1}$) denote the inverse of $\varphi(\omega, t)$ (resp., $\varphi^*(\omega, t)$) with respect to t, i.e. $t = \varphi^{-1}(\varphi(\omega, t))$. Then for all $t \geqslant 0$

$$t \leqslant (\varphi^{-1})(t)\bigl((\varphi^*)^{-1}\bigr)(t) \leqslant 2t, \tag{2.2}$$

$$\frac{t}{2}\varphi'\left(\frac{t}{2}\right) \leqslant \varphi(t) \leqslant t\varphi'(t), \tag{2.3}$$

$$\varphi\left(\frac{\varphi^*(t)}{t}\right) \leqslant \varphi^*(t) \leqslant \varphi\left(\frac{2\varphi^*(t)}{t}\right), \tag{2.4}$$

where we skipped the dependence on ω to make the inequalities better to read. If $\psi(\omega, t) = a\varphi(\omega, bt)$ for some $a, b > 0$, then

$$\psi^*(\omega, t) = a\varphi^*\left(\omega, \frac{t}{ab}\right). \tag{2.5}$$

If φ and ψ are N-function with $\varphi(\omega, t) \leqslant \psi(\omega, t)$ for all $t \geqslant 0$, then

$$\psi^*(\omega, t) \leqslant \varphi^*(\omega, t) \tag{2.6}$$

for all $t \geqslant 0$.

Definition 2.2. We say that φ is a proper N-function on $\Omega = \mathbb{R}^d$ (resp., $\Omega = \mathcal{X}^d$) if φ is an N-function and φ and φ^* satisfy the strong Δ_2-condition.

Note that if φ is a proper N-function then by Section 13 of [22] there follows $(L^\varphi)^*(\mathbb{R}^d) \cong L^{\varphi^*}(\mathbb{R}^d)$ and $(L^{\varphi^*})^*(\mathbb{R}^d) \cong L^\varphi(\mathbb{R}^d)$.

Example 2.3. Let $p:\mathbb{R}^d \to [1,\infty)$ be measurable with $1 < p^- \leqslant p^+ < \infty$ where $p^- := \inf p$, $p^+ := \sup p$. The function p is called a bounded exponent on $\mathbb{R}^d$ (bounded, since $p^+ < \infty$). Define $p':\mathbb{R}^d \to [1,\infty)$ by $1 = \frac{1}{p(x)} + \frac{1}{p'(x)}$. Then $\varphi(x,t) := t^{p(x)}$ is a proper N-function on $\mathbb{R}^d$ and its complementary function φ^* is given by $\varphi^*(x,t) = (p(x)-1) \times p(x)^{-p'(x)} t^{p'(x)}$. The space $L^\varphi(\mathbb{R}^d)$ is denoted by $L^{p(\cdot)}(\mathbb{R}^d)$. Since $(p(x)-1)p(x)^{-p'(x)}$ is bounded from above and below (away from zero), it follows that $(L^{p(\cdot)})^*(\mathbb{R}^d) \cong L^{\varphi^*} \cong L^{p'(\cdot)}(\mathbb{R}^d)$. Note that sometimes it is more convenient to work with $(x,t) \mapsto \frac{1}{p(x)} t^{p(x)}$, since its complementary function is $(x,t) \mapsto \frac{1}{p'(x)} t^{p'(x)}$. This defines up to isomorphism the same space, which is also called generalized Lebesgue space, Lebesgue space with variable exponent, or $L^{p(x)}(\mathbb{R}^d)$. For an introduction to $L^{p(\cdot)}(\mathbb{R}^d)$ spaces see [19] and [15]. If p is constant then $L^{p(\cdot)}(\mathbb{R}^d)$ is isomorphic to the classical Lebesgue space $L^p(\mathbb{R}^d)$.

Example 2.4. Let φ be a proper N-function. For a weight ω on $\mathbb{R}^d$, i.e. a positive, measurable function on $\mathbb{R}^d$, we define $\varphi_\omega(x,t) := \varphi(x,\omega(x)t)$. Then φ_ω is a proper N-function on $\mathbb{R}^d$ with $(\varphi_\omega)^*(x,t) = (\varphi^*)_{1/\omega}(x,t)$. Define $L^\varphi_\omega(\mathbb{R}^d) := L^{\varphi_\omega}(\mathbb{R}^d)$, then $\|f\|_{L^\varphi_\omega} = \|f\|_{\varphi_\omega} = \|f\omega\|_\varphi$. (This is consistent with the usual definition of X_ω for a Banach function space X.) Moreover, holds $(L^\varphi_\omega(\mathbb{R}^d))^* \cong L^{\varphi^*}_{1/\omega}(\mathbb{R}^d)$. Let $L^{p(\cdot)}_\omega(\mathbb{R}^d) := L^{\varphi_p}_\omega(\mathbb{R}^d)$, where $\varphi_p(x,t) := t^{p(x)}$ as in Example 2.3, then $(L^{p(\cdot)}_\omega(\mathbb{R}^d))^* \cong L^{p'(\cdot)}_{1/\omega}(\mathbb{R}^d)$. Note that $L^{p(\cdot)}_\omega(\mathbb{R}^d)$ also corresponds to the Musielak–Orlicz space $L^{p(\cdot)}(\mathbb{R}^d, \mathrm{d}(\omega^p))$, i.e. the Lebesgue measure $\mathrm{d}x$ is replaced by $\omega(x)^{p(x)}\,\mathrm{d}x$ (see [22]). If p is constant then $L^{p(\cdot)}_\omega(\mathbb{R}^d)$ is isomorphic to the classical weighted Lebesgue space $L^p_\omega(\mathbb{R}^d) = L^p(\mathbb{R}^d, \mathrm{d}\omega^p)$.

3. Class $\mathcal{A}$

For $Q \in \mathcal{X}^d$, $s \in [1,\infty)$, and $f \in L^s_{\mathrm{loc}}$, we define

$$M_{s,Q} f := \left(\fint_Q |f(x)|^s \,\mathrm{d}x \right)^{1/s} \equiv \left(\frac{1}{|Q|} \int_Q |f(x)|^s \,\mathrm{d}x \right)^{1/s},$$

$$M_Q f := M_{1,Q} f.$$

We define the maximal function $M_s f:\mathbb{R}^d \to [0,\infty]$ by

$$(M_s f)(x) := \sup_{Q \ni x} M_{s,Q} f, \qquad Mf := M_1 f,$$

where the supremum is taken over all cubes $Q \in \mathcal{X}^d$ containing $x \in \mathbb{R}^d$. For $\mathcal{Q} \in \mathcal{Y}^d$ and $f \in L^1_{\mathrm{loc}}$ we define $T_{\mathcal{Q}}: L^1_{\mathrm{loc}} \to \mathcal{F}^d$ by

$$T_{\mathcal{Q}} f := \sum_{Q \in \mathcal{Q}} \chi_Q M_Q f.$$

We will now generalize the concept of Muckenhoupt classes to Musielak–Orlicz spaces.

Definition 3.1. Let φ be a proper N-function on $\mathbb{R}^d$. We say that φ is of class $\mathcal{A}$ if and only if there exists $C_3 > 0$ such that for all $\mathcal{Q} \in \mathcal{Y}^d$ and all $f \in L^\varphi(\mathbb{R}^d)$

$$\|T_\mathcal{Q} f\|_\varphi \equiv \left\| \sum_{Q \in \mathcal{Q}} \chi_Q M_Q f \right\|_\varphi \leqslant C_3 \|f\|_\varphi,$$

i.e. the averaging operators $T_\mathcal{Q}$ are uniformly continuous on $L^\varphi(\mathbb{R}^d)$ with respect to $\mathcal{Q} \in \mathcal{Y}^d$.

In the case of weighted (classical) Lebesgue spaces this definition coincides with the Muckenhoupt classes A_q, i.e. if $\varphi(x,t) = t^q \omega(x)$, $1 < q < \infty$, then φ is of class $\mathcal{A}$ if and only if $\omega \in A_q$. Definition 3.1 also generalizes a condition given by Berezhnoi for ideal Banach spaces, see [2] Definition 2, which is stated for single cubes only and not for families of disjoint cubes. Berezhnoi further studies spaces with the property $\mathbf{G}(\mathbf{B})$ which basically says that his condition stated for single cubes can be transfered to families of disjoint cubes. Nevertheless, Musielak–Orlicz spaces fail in general property $\mathbf{G}(\mathbf{B})$. Recently, Kopaliani [18] has studied a similar condition $G(\Pi_*)$ on $L^{p(\cdot)}([0,1])$. Kopaliani shows that if p satisfies the uniform local condition $|p(x) - p(y)| \leqslant C|\ln|x-y||^{-1}$ then $L^{p(\cdot)}([0,1])$ has condition $G(\Pi_*)$, i.e. it is sufficient to consider single cubes in Definition 3.1.

Lemma 3.2. *Let φ be a proper N-function on $\mathbb{R}^d$ such that M is continuous on $L^\varphi(\mathbb{R}^d)$. Then φ is of class $\mathcal{A}$.*

Proof. Let $f \in L^\varphi(\mathbb{R}^d)$ then $T_\mathcal{Q} f \leqslant Mf$. So $\|T_\mathcal{Q} f\|_\varphi \leqslant \|Mf\|_\varphi \leqslant C\|f\|_\varphi$, where C is independent of $\mathcal{Q}$. This proves the lemma. $\square$

In order to characterize proper N-functions which are of class $\mathcal{A}$, we need more notation. Let φ be a proper N-function on $\mathbb{R}^d$. For $t \geqslant 0$, $s \geqslant 1$, and $f \in \mathcal{F}^d$ we define

$$\varphi(f) : \mathbb{R}^d \to \mathbb{R}^{\geqslant 0}, \quad \big(\varphi(f)\big)(x) := \varphi\big(x, |f(x)|\big),$$
$$\varphi(t) : \mathbb{R}^d \to \mathbb{R}^{\geqslant 0}, \quad \big(\varphi(t)\big)(x) := \varphi(x,t),$$
$$M_{s,Q}\varphi : R \to \mathbb{R}^{\geqslant 0}, \quad (M_{s,Q}\varphi)(t) := M_{s,Q}\big(\varphi(t)\big) \equiv \left(\fint_Q \big(\varphi(x,t)\big)^s \,\mathrm{d}x \right)^{1/s},$$
$$M_Q\varphi : R \to \mathbb{R}^{\geqslant 0}, \quad (M_Q\varphi)(t) := (M_{1,Q}\varphi)(t).$$

Analogously, we define for a proper N-function φ on $\mathcal{X}^d$

$$\varphi(Q) : \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}, \quad \big(\varphi(Q)\big)(t) := \varphi(Q,t).$$

Lemma 3.3. *Let φ be a proper N-function $\mathbb{R}^d$. Then φ is of class $\mathcal{A}$ if and only if φ^* is of class $\mathcal{A}$.*

Proof. Since φ is a proper N-function, there holds $(L^\varphi(\mathbb{R}^d))^* \cong L^{\varphi^*}(\mathbb{R}^d)$ and $(L^{\varphi^*}(\mathbb{R}^d))^* \cong L^\varphi(\mathbb{R}^d)$ (see Section 13 of [22]). Thus

$$\begin{aligned}\|T_{\mathcal{Q}}\|_{\varphi^*\to\varphi^*} &\sim \sup_{\|g\|_{\varphi^*}\leqslant 1}\sup_{\|f\|_\varphi\leqslant 1}\langle T_{\mathcal{Q}}g, f\rangle \\ &= \sup_{\|g\|_{\varphi^*}\leqslant 1}\sup_{\|f\|_\varphi\leqslant 1}\big\langle T_{\mathcal{Q}}|g|, |f|\big\rangle = \sup_{\|g\|_{\varphi^*}\leqslant 1}\sup_{\|f\|_\varphi\leqslant 1}\big\langle |g|, T_{\mathcal{Q}}|f|\big\rangle \\ &= \sup_{\|f\|_\varphi\leqslant 1}\sup_{\|g\|_{\varphi^*}\leqslant 1}\langle g, T_{\mathcal{Q}}f\rangle \sim \|T_{\mathcal{Q}}\|_{\varphi\to\varphi}.\end{aligned}$$

Hence φ is of class $\mathcal{A}$ if and only if φ^* is of class $\mathcal{A}$. $\square$

Let φ be a proper N-function on $\mathcal{X}^d$. Let $\mathcal{Q} \in \mathcal{Y}^d$, then $\mathcal{Q}$ is countable (including finite) and can therefore be identified by a subset on N. Thus φ can be interpreted as a proper N-function on $\mathcal{Q}$. This enables us to define the Musielak–Orlicz sequence space $l^{|Q|\varphi(Q)}$ by

$$l^{|Q|\varphi(Q)} := \left\{\bar{t} = \{t_Q\}_{Q\in\mathcal{Q}} \in \mathbb{R}^{\mathcal{Q}}\colon \sum_{Q\in\mathcal{Q}} |Q|\varphi\big(Q, |t_Q|\big) < \infty\right\},$$

equipped with the norm

$$\|\bar{t}\|_{l^{|Q|\varphi(Q)}} := \inf\left\{\lambda > 0\colon \sum_{Q\in\mathcal{Q}} |Q|\varphi\big(Q, |t_Q/\lambda|\big) < 1\right\}.$$

From the theory of Musielak–Orlicz spaces we deduce that $l^{|Q|\varphi(Q)}$ is a Banach space (see [22]).

Lemma 3.4. *Let φ be a proper N-function on $\mathbb{R}^d$ and $s \geqslant 1$, then $(Q,t) \mapsto (M_{s,Q}\varphi)(t)$ is a proper N-function on $\mathcal{X}^d$.*

Proof. It is easy to see that $(Q,t) \mapsto (M_{s,Q}\varphi)(t)$ is an N-function on $\mathcal{X}^d$ which satisfies the strong Δ_2-condition (see e.g. Definition 13.1 in [22] for an alternative characterization of N-functions). It remains to show that $(M_{s,Q}\varphi)^*(t)$ satisfies the strong Δ_2-condition. Since φ^* satisfies the strong Δ_2-condition there exists $A \geqslant 2$ with $\varphi^*(x,2t) \leqslant A\varphi^*(x,t)$ for all $x \in \mathbb{R}^d$ and $t > 0$. Due to (2.5) and (2.6) this is equivalent to $\varphi(x,t/2) \geqslant A\varphi(t/A)$ for all $x \in \mathbb{R}^d$ and $t > 0$. It follows that $(M_{s,Q}\varphi)(x,t/2) \geqslant A(M_{s,Q}\varphi)(t/A)$ for all $Q \in \mathcal{X}^d$ and $t > 0$. Again due to (2.5) and (2.6) this is equivalent to $(M_{s,Q}\varphi)^*(x,2t) \leqslant A(M_{s,Q}\varphi)^*(t)$ for all $Q \in \mathcal{X}^d$ and $t > 0$. This proves the lemma. $\square$

Let φ be a proper N-function on $\mathbb{R}^d$. Then due to Lemma 3.4 we can define the space $l^{|Q|M_Q\varphi}$ in the sense above, i.e.

$$l^{|Q|M_Q\varphi} := \left\{\bar{t} = \{t_Q\}_{Q\in\mathcal{Q}} \in \mathbb{R}^{\mathcal{Q}}\colon \sum_{Q\in\mathcal{Q}} |Q|(M_Q\varphi)(t_Q) < \infty\right\},$$

equipped with the norm

$$\|\bar{t}\|_{l^{|Q|M_Q\varphi}} := \inf\left\{\lambda > 0\colon \sum_{Q\in\mathcal{Q}} |Q|(M_Q\varphi)\left(\frac{|t_Q|}{\lambda}\right) < 1\right\}.$$

On the other hand φ^* is also a proper N-function on $\mathbb{R}^d$, so $(Q,t) \mapsto M_Q\varphi^*$ is proper N-function on $\mathcal{X}^d$. Thus there exists the complementary function $(M_Q\varphi^*)^*$ (again a proper N-function on $\mathcal{X}^d$). Especially, we will consider the space $l^{|Q|(M_Q\varphi^*)^*}$.

Definition 3.5. Let φ, ψ be proper N-functions on $\mathcal{X}^d$ such that

$$l^{|Q|\varphi(Q)}(\mathcal{Q}) \hookrightarrow l^{|Q|\psi(Q)}(\mathcal{Q})$$

are uniformly continuous with respect to $\mathcal{Q} \in \mathcal{Y}^d$, i.e. for all $A_1 > 0$ there exists $A_2 > 0$ such that for all $\mathcal{Q} \in \mathcal{Y}^d$ and all sequences $\{t_Q\}_{Q\in\mathcal{Q}}$, $t_Q \in \mathbb{R}^{\geqslant 0}$, there holds

$$\sum_{Q\in\mathcal{Q}} |Q|\varphi(Q,t_Q) \leqslant A_1 \quad \Rightarrow \quad \sum_{Q\in\mathcal{Q}} |Q|\psi(Q,t_Q) \leqslant A_2. \tag{3.1}$$

Then we say that ψ is dominated by φ and write $\psi \lesssim \varphi$, $\psi(Q) \lesssim \varphi(Q)$, or $\psi(Q,t) \lesssim \varphi(Q,t)$.

Note that due to the strong Δ_2-condition is suffices verify (3.1) for one couple $A_1, A_2 > 0$. We can now state our alternative characterization of class $\mathcal{A}$.

Theorem 3.6. *Let φ be a proper N-function on $\mathbb{R}^d$. Then φ is of class $\mathcal{A}$ if and only if $M_Q\varphi \lesssim (M_Q\varphi^*)^*$.*

Before we get to the proof of Theorem 3.6 we need the following lemma.

Lemma 3.7. *Let φ be a proper N-function on $\mathbb{R}^d$. Let $s \geqslant 1$ and let $Q \in \mathcal{X}^d$. Then for all $f \in L^{\varphi}(Q)$ there holds*

$$(M_{s,Q}\varphi^*)^*\left(\frac{1}{2}M_{s,Q}f\right) \leqslant M_{s,Q}\big(\varphi(f)\big). \tag{3.2}$$

Especially, for all $u \geqslant 0$

$$(M_{s,Q}\varphi^*)^*\left(\frac{1}{2}u\right) \leqslant (M_{s,Q}\varphi)(u). \tag{3.3}$$

On the other hand for all $t > 0$ the function $f_t := \chi_Q\varphi^*(t)/t$ satisfies

$$(M_{s,Q}\varphi^*)^*(2M_{s,Q}f_t) \geqslant M_{s,Q}\big(\varphi(f_t)\big). \tag{3.4}$$

Proof. For $f \in L^{\varphi}(Q)$ and $f \not\equiv 0$ define

$$\lambda := \frac{1}{2}M_{s,Q}f, \qquad \kappa := \frac{(M_{s,Q}\varphi^*)^*(\lambda)}{\lambda}.$$

From (2.4) we deduce

$$\begin{aligned}(M_{s,Q}\varphi^*)(\kappa) &= (M_{s,Q}\varphi^*)\left(\frac{(M_{s,Q}\varphi^*)^*(\lambda)}{\lambda}\right)\\ &\leqslant (M_{s,Q}\varphi^*)^*(\lambda) \quad \text{by (2.4)}\\ &= \lambda\kappa.\end{aligned} \tag{3.5}$$

On the other hand by (2.1)

$$2\lambda\kappa = M_{s,Q}(f\kappa) \leqslant M_{s,Q}\big(\varphi(f)\big) + (M_{s,Q}\varphi^*)(\kappa).$$

This and (3.5) implies $\lambda\kappa \leqslant M_{s,Q}(\varphi(f))$. Hence

$$(M_{s,Q}\varphi^*)^*\left(\frac{1}{2}M_{s,Q}f\right) = (M_{s,Q}\varphi^*)^*(\lambda) = \lambda\kappa \leqslant M_{s,Q}\big(\varphi(f)\big).$$

This proves (3.2), while (3.3) follows from $f := \chi_Q u$.

Let $t > 0$ and $f_t := \chi_Q\varphi^*(t)/t$, then

$$\begin{aligned}(M_{s,Q}\varphi^*)^*(2M_{s,Q}f_t) &= (M_{s,Q}\varphi^*)^*\left(\frac{2(M_{s,Q}\varphi^*)(t)}{t}\right)\\ &\geqslant (M_{s,Q}\varphi^*)(t) \quad \text{by (2.4)}\\ &\geqslant M_{s,Q}\left(\varphi\left(\frac{\varphi^*(t)}{t}\right)\right) \quad \text{by (2.4)}\\ &= M_{s,Q}\big(\varphi(f_t)\big).\end{aligned}$$

This proves (3.4).

We are now prepared to prove Theorem 3.6.

Proof of Theorem 3.6. *Case* (b) $\Rightarrow$ (a): Let $f \in L^\varphi(\mathbb{R}^d)$ with $\|f\|_\varphi \leqslant 1$. Let $\mathcal{Q} \in \mathcal{Y}^d$, then

$$\sum_{Q\in\mathcal{Q}} |Q| M_Q\big(\varphi(f)\big) \leqslant \int_{\mathbb{R}^d} \varphi(f)\,\mathrm{d}x \leqslant 1.$$

Due to Lemma 3.7 holds

$$\sum_{Q\in\mathcal{Q}} |Q|(M_Q\varphi^*)^*\left(\frac{1}{2}M_Q f\right) \leqslant 1. \tag{3.6}$$

By assumption there exists $A > 0$ (independent of $\mathcal{Q}$) such that (3.6) implies

$$\int_{\mathbb{R}^d} \varphi\left(\frac{1}{2}\sum_{Q\in\mathcal{Q}} \chi_Q M_Q f\right)\mathrm{d}x = \sum_{Q\in\mathcal{Q}} |Q|(M_Q\varphi)\left(\frac{1}{2}M_Q f\right) \leqslant A.$$

Since φ satisfies the strong Δ_2-condition, there exists A_2 (independent of $\mathcal{Q}$) such that

$$\left\|\sum_{Q\in\mathcal{Q}} \chi_Q M_Q f\right\|_\varphi \leqslant A_2.$$

This proves (a).

Case (a) $\Rightarrow$ (b): Let $\mathcal{Q} \in \mathcal{Y}^d$ and $\bar{t} = \{t_Q\}_{Q\in\mathcal{Q}} \in l^{|Q|(M_Q\varphi^*)^*}$ with

$$\sum_{Q\in\mathcal{Q}} |Q|(M_Q\varphi^*)^*(t_Q) \leqslant 1.$$

Define $\bar{u} = \{u_Q\}_{Q\in\mathcal{Q}}$ such that

$$t_Q = 2M_Q\bigl(\varphi^*(u_Q)\bigr)/u_Q.$$

Since φ is a proper N-function, $\bar{u}$ is well defined and unique. Define $f \in \mathcal{F}^d$ by

$$f := \sum_{Q\in\mathcal{Q}} \chi_Q \frac{\varphi^*(u_Q)}{u_Q},$$

then $2M_Q f = t_Q$ for all $Q \in \mathcal{Q}$. Moreover, (3.4) implies

$$(M_Q\varphi^*)^*(t_Q) = (M_Q\varphi^*)^*(2M_Q f) \geqslant M_Q\bigl(\varphi(f)\bigr).$$

This implies

$$\int_{\mathbb{R}^d} \varphi(f)\,\mathrm{d}x = \sum_{Q\in\mathcal{Q}} |Q| M_Q\bigl(\varphi(f)\bigr) \leqslant \sum_{Q\in\mathcal{Q}} |Q| (M_Q\varphi^*)^*(t_Q) \leqslant 1,$$

especially $\|f\|_\varphi \leqslant 1$. Since φ is of class $\mathcal{A}$, there exists $A_3 > 0$ (independent of $\mathcal{Q}$) such that

$$\Bigl\| \sum_{Q\in\mathcal{Q}} \chi_Q M_Q f \Bigr\|_\varphi \leqslant A_3.$$

Since φ satisfies the strong Δ_2-condition, there exists $A_4 > 0$ (independent of $\mathcal{Q}$) such that

$$\sum_{Q\in\mathcal{Q}} |Q| (M_Q\varphi)(M_Q f) = \int_{\mathbb{R}^d} \varphi\Bigl(\sum_{Q\in\mathcal{Q}} \chi_Q M_Q f \Bigr)\,\mathrm{d}x \leqslant A_4.$$

The definition of f implies

$$\sum_{Q\in\mathcal{Q}} |Q| (M_Q\varphi)\Bigl(\frac{1}{2} t_Q\Bigr) = \sum_{Q\in\mathcal{Q}} |Q| (M_Q\varphi)(M_Q f) \leqslant A_4.$$

This and the strong Δ_2-condition prove (b). $\square$

Let us give some remark on complementary functions.

Lemma 3.8. *If φ and ψ are proper N-functions on $\mathcal{X}^d$. If $\psi \lesssim \varphi$ then $\varphi^* \lesssim \psi^*$.*

Proof. Let $\psi \lesssim \varphi$, then $l^{|Q|\varphi(Q)}(\mathcal{Q}) \hookrightarrow l^{|Q|\psi(Q)}(\mathcal{Q})$ uniformly in $\mathcal{Q} \in \mathcal{Y}^d$. Since φ and ψ are proper, there follows by duality (see Section 13 of [22]) $l^{(|Q|\psi(Q))^*}(\mathcal{Q}) \hookrightarrow l^{(|Q|\varphi(Q))^*}(\mathcal{Q})$ uniformly in $\mathcal{Q} \in \mathcal{Y}^d$, i.e. there exists $A > 0$ (independent on $\mathcal{Q}$) such that for all sequences $\{t_Q\}_{Q\in\mathcal{Q}}$, $t_Q \geqslant 0$,

$$\sum_{Q\in\mathcal{Q}} \bigl(|Q|\psi(Q)\bigr)^*(t_Q) \leqslant 1 \quad \Rightarrow \quad \sum_{Q\in\mathcal{Q}} \bigl(|Q|\varphi(Q)\bigr)^*(t_Q) \leqslant A. \tag{3.7}$$

Due to (2.5) there holds

$$\bigl(|Q|\varphi(Q)\bigr)^*(t) = |Q|\varphi^*\Bigl(Q, \frac{t}{|Q|}\Bigr), \qquad \bigl(|Q|\psi(Q)\bigr)^*(t) = |Q|\psi^*\Bigl(Q, \frac{t}{|Q|}\Bigr). \tag{3.8}$$

Thus applying (3.7) to the sequence $\{t_Q|Q|\}_{Q\in\mathcal{Q}}$ and (3.8) imply

$$\sum_{Q\in\mathcal{Q}} |Q|\psi^*(Q,t_Q) \leqslant 1 \quad \Rightarrow \quad \sum_{Q\in\mathcal{Q}} |Q|\varphi^*(Q,t_Q) \leqslant A.$$

This proves $\varphi^* \lesssim \psi^*$. □

4. Class $\mathcal{A}$ for generalized Lebesgue spaces

In the case of generalized Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^d)$ we can provide an alternative characterization of class $\mathcal{A}$. Let p be a bounded exponent on $\mathbb{R}^d$ with $1 < p^- \leqslant p^+ < \infty$ and let $\varphi(x,t) = t^{p(x)}$ for all $t \in \mathbb{R}^{\geqslant 0}$ and all $x \in \mathbb{R}^d$ (see Example 2.3). For $Q \in \mathcal{X}^d$ we define $\bar{p}_Q$ by $\frac{1}{\bar{p}_Q} := \fint_Q \frac{1}{p}\,\mathrm{d}x$. Then $(\bar{p}_Q)' = \bar{p}'_Q$, where $\frac{1}{p'} + \frac{1}{p} = 1$, we therefore simply write $\bar{p}'_Q$.

Lemma 4.1. *Let p be a bounded exponent on $\mathbb{R}^d$ with $1 < p^- \leqslant p^+ < \infty$ and let $\varphi(x,t) := t^{p(x)}$, then for all $t \geqslant 0$ and all $Q \in \mathcal{X}^d$*

$$(M_Q\varphi)(t) = M_Q(t^p) \geqslant \mathrm{e}^{2(p^- - p^+)} t^{\bar{p}_Q}, \tag{4.1}$$

$$(M_Q\varphi^*)(t) \geqslant M_Q(t^{p'}) \geqslant \mathrm{e}^{2((p^+)' - (p^-)')} t^{\bar{p}'_Q}. \tag{4.2}$$

Especially there holds uniformly in $\mathcal{Q}$

$$t^{\bar{p}_Q} \lesssim M_Q\varphi, \qquad t^{\bar{p}'_Q} \lesssim M_Q\varphi^*, \tag{4.3}$$

$$(M_Q\varphi)^* \lesssim t^{\bar{p}'_Q}, \qquad (M_Q\varphi^*)^* \lesssim t^{\bar{p}_Q}. \tag{4.4}$$

Proof. The case $t = 0$ is obvious, so assume $t > 0$. Define $f_t : (0,1] \to \mathbb{R}^{>0}$, $u \mapsto t^{1/u}$, then $f''_t(u) = t^{1/u}u^{-4}(\ln t)((\ln t) + 2u)$. If $0 < t < \mathrm{e}^{-2}$ or $t \geqslant 1$ then f_t is convex and by Jensen's inequality there follows

$$M_Q(t^p) = \fint_Q f_t\left(\frac{1}{p}\right)\mathrm{d}x \geqslant f_t\left(\fint_Q \frac{1}{p}\,\mathrm{d}x\right) = t^{\bar{p}_Q}. \tag{4.5}$$

Assume now that $\mathrm{e}^{-2} \leqslant t \leqslant 1$, then

$$M_Q(t^p) \geqslant \mathrm{e}^{2p^-} M_Q\big((\mathrm{e}^{-2}t)^p\big) \overset{(4.5)}{\geqslant} \mathrm{e}^{2p^-}(\mathrm{e}^{-2}t)^{\bar{p}_Q} \geqslant \mathrm{e}^{2(p^- - p^+)} t^{\bar{p}_Q}.$$

This proves (4.1). Since $\varphi^*(x,t) = (p(x) - 1)p(x)^{-p'(x)}t^{p'(x)}$ (see Example 2.3) and $(p(x)-1)p(x)^{-p'(x)} = (\frac{1}{p'(x)}p(x)^{1/(1-p(x))}) \leqslant 1$, there holds $\varphi^*(x,t) \leqslant t^{p'(x)}$. Now (4.2) follows from (4.1) applied to p' and from $\bar{p}'_Q = \bar{p}'_Q$, $(p')^- = (p^+)'$, $(p')^+ = (p^-)'$. Moreover, (4.3) is a direct consequence of (4.1) and (4.2). Since $(t^{\bar{p}_Q})^* \sim t^{\bar{p}'_Q}$ and $(t^{\bar{p}'_Q})^* \sim t^{\bar{p}_Q}$, Lemma 3.8 and (4.3) prove (4.4). □

Theorem 4.2. *Let p be a bounded exponent on $\mathbb{R}^d$ with $1 < p^- \leqslant p^+ < \infty$. Let $\varphi(x,t) := t^{p(x)}$. Then the following conditions are equivalent*:

(a) *φ is of class $\mathcal{A}$.*
(b) $M_Q\varphi \lesssim (M_Q\varphi^*)^*$.
(c) $M_Q\varphi \lesssim t^{\bar{p}_Q}$ *and* $M_Q\varphi^* \lesssim t^{\bar{p}'_Q}$.

Proof. Note that $(M_Q\varphi^*)^*(t) \lesssim t^{\bar{p}_Q} \lesssim (M_Q\varphi)(t)$ by Lemma 4.1.

Case (a) $\Leftrightarrow$ (b): Follows by Theorem 3.6.

Case (b) $\Rightarrow$ (c): Let φ be of class $\mathcal{A}$, then $t^{\bar{p}_Q} \lesssim M_Q\varphi \lesssim (M_Q\varphi^*)^* \lesssim t^{\bar{p}_Q}$. Thus $t^{\bar{p}_Q} \lesssim (M_Q\varphi^*)^*$ and $M_Q\varphi \lesssim t^{\bar{p}_Q}$. With Lemma 3.8 we deduce $t^{\bar{p}'_Q} \lesssim M_Q\varphi^*$.

Case (c) $\Rightarrow$ (b): Let $M_Q\varphi \lesssim t^{\bar{p}_Q}$ and $M_Q\varphi^* \lesssim t^{\bar{p}'_Q}$. Then by Lemma 3.8 $t^{\bar{p}_Q} \lesssim (M_Q\varphi^*)^*$. Thus $M_Q\varphi \lesssim t^{\bar{p}_Q} \lesssim (M_Q\varphi^*)^*$. Especially, $M_Q\varphi \lesssim (M_Q\varphi^*)^*$. □

We will get back to the spaces $L^{p(\cdot)}(\mathbb{R}^d)$ in Section 8.

5. Class $\mathcal{A}_\infty$

In this section we will define an analogy of the Muckenhoupt class A_∞. We will see that as in the case of Muckenhoupt class A_∞ our new condition $\mathcal{A}_\infty$ will imply an improvement of integrability, i.e. we will show in Theorem 5.6 that $\mathcal{A}_\infty$ implies $M_{s,Q}\varphi \lesssim M_Q\varphi$ for some $s > 1$. This will be our substitute for the reverse Hölder estimate of (classical) Muckenhoupt weights.

Definition 5.1. Let φ be a proper N-function on $\mathbb{R}^d$. We say that φ is of class $\mathcal{A}_\infty$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that the following holds: If $N \subset \mathbb{R}^d$ is measurable and $\mathcal{Q} \in \mathcal{Y}^d$ such that

$$|Q \cap N| \geqslant \varepsilon |Q| \quad \text{for all } Q \in \mathcal{Q}, \tag{5.1}$$

then for any sequence $\{t_Q\}_{Q\in\mathcal{Q}}$, $t_Q \in \mathbb{R}^{\geqslant 0}$

$$\delta \Bigl\| \sum_{Q\in\mathcal{Q}} t_Q \chi_Q \Bigr\|_\varphi \leqslant \Bigl\| \sum_{Q\in\mathcal{Q}} t_Q \chi_{Q\cap N} \Bigr\|_\varphi. \tag{5.2}$$

Note that in the case of weighted (classical) Lebesgue spaces this definition coincides with the Muckenhoupt class A_∞, i.e. $\varphi(x,t) = t^q \omega(x)$, $1 < q < \infty$, is of class $\mathcal{A}_\infty$ if and only if $\omega \in A_\infty$.

Lemma 5.2. *Let φ be a proper N-function on $\mathbb{R}^d$ of class $\mathcal{A}$. Then φ is of class $\mathcal{A}_\infty$.*

Proof. Let $\varepsilon > 0$ and $\mathcal{Q}$, N as in (5.1). Let $f := \sum_{Q\in\mathcal{Q}} s_Q \chi_{Q\cap N}$ with $\|f\|_\varphi < \infty$. Then

$$\varepsilon \sum_{Q\in\mathcal{Q}} s_Q \chi_Q \leqslant \sum_{Q\in\mathcal{Q}} s_Q \frac{|Q\cap N|}{|Q|} \chi_Q = \sum_{Q\in\mathcal{Q}} (M_Q f)\chi_Q.$$

Since φ is of class $\mathcal{A}$ there follows

$$\varepsilon \biggl\| \sum_{Q \in \mathcal{Q}} s_Q \chi_Q \biggr\|_\varphi \leqslant \biggl\| \sum_{Q \in \mathcal{Q}} (M_Q f) \chi_Q \biggr\|_\varphi \leqslant C \|f\|_\varphi.$$

This proves (5.2) with $\delta := \varepsilon / C$. Thus φ is of class $\mathcal{A}_\infty$. $\square$

Lemma 5.3. *Let φ be a proper N-function on $\mathbb{R}^d$ of class $\mathcal{A}_\infty$. Then for every $0 < \alpha < 1$ there exists $0 < \beta < 1$ such that the following holds: If $N \subset \mathbb{R}^d$ is measurable and $\mathcal{Q} \in \mathcal{Y}^d$ such that*

$$|N \cap Q| \leqslant \alpha |Q| \quad \textit{for all } Q \in \mathcal{Q}, \tag{5.3}$$

then for any sequence $\{s_Q\}_{Q \in \mathcal{Q}}$, $s_Q \in \mathbb{R}^{\geqslant 0}$ holds

$$\biggl\| \sum_{Q \in \mathcal{Q}} s_Q \chi_{N \cap Q} \biggr\|_\varphi \leqslant \beta \biggl\| \sum_{Q \in \mathcal{Q}} s_Q \chi_Q \biggr\|_\varphi. \tag{5.4}$$

Proof. Define $P := \mathbb{R}^d \setminus (\bigcup_{Q \in \mathcal{Q}} Q)$. From (5.3) we deduce

$$|Q \cap P| = |Q \setminus N| \geqslant (1 - \alpha)|Q| \quad \text{for all } Q \in \mathcal{Q}.$$

Let $\{s_Q\}_{Q \in \mathcal{Q}}$, $s_Q \in \mathbb{R}^{\geqslant 0}$. Since φ is of class $\mathcal{A}_\infty$ there exists $\delta > 0$ such that

$$\delta \biggl\| \sum_{Q \in \mathcal{Q}} s_Q \chi_Q \biggr\|_\varphi \leqslant \biggl\| \sum_{Q \in \mathcal{Q}} s_Q \chi_{P \cap Q} \biggr\|_\varphi.$$

Assume without loss of generality that $\{s_Q\}_{Q \in \mathcal{Q}} \neq \{0\}_{Q \in \mathcal{Q}}$, so

$$\biggl\| \frac{\sum_{Q \in \mathcal{Q}} s_Q \chi_{P \cap Q}}{\bigl\| \sum_{Q \in \mathcal{Q}} s_Q \chi_Q \bigr\|_\varphi} \biggr\|_\varphi \geqslant \delta.$$

Since φ satisfies the strong Δ_2-condition there exists $a > 0$ with

$$\int \varphi\biggl(\frac{\sum_{Q \in \mathcal{Q}} s_Q \chi_{P \cap Q}}{\bigl\| \sum_{Q \in \mathcal{Q}} s_Q \chi_Q \bigr\|_\varphi} \biggr) \geqslant a.$$

Thus

$$\begin{aligned}
&\int \varphi\biggl(\frac{\sum_{Q \in \mathcal{Q}} s_Q \chi_{N \cap Q}}{\bigl\| \sum_{Q \in \mathcal{Q}} s_Q \chi_Q \bigr\|_\varphi} \biggr) \\
&\quad = \int \varphi\biggl(\frac{\sum_{Q \in \mathcal{Q}} s_Q \chi_Q}{\bigl\| \sum_{Q \in \mathcal{Q}} s_Q \chi_Q \bigr\|_\varphi} \biggr) - \int \varphi\biggl(\frac{\sum_{Q \in \mathcal{Q}} s_Q \chi_{Q \setminus N}}{\| \sum_{Q \in \mathcal{Q}} s_Q \chi_Q \|_\varphi} \biggr) \\
&\quad = 1 - \int \varphi\biggl(\frac{\sum_{Q \in \mathcal{Q}} s_Q \chi_{P \cap Q}}{\bigl\| \sum_{Q \in \mathcal{Q}} s_Q \chi_Q \bigr\|_\varphi} \biggr) \leqslant 1 - a.
\end{aligned}$$

Since φ satisfies the strong Δ_2-condition there exists $0 < \beta < 1$ with

$$\biggl\| \frac{\sum_{Q \in \mathcal{Q}} s_Q \chi_{N \cap Q}}{\| \sum_{Q \in \mathcal{Q}} s_Q \chi_Q \|_\varphi} \biggr\| \leqslant \beta.$$

This proves the lemma. $\square$

For the proof of the next lemma it is convenient to work with dyadic cubes.

Definition 5.4. We say that $Q \in \mathcal{X}^d$ is *dyadic* if there exists $\bar{k} = (k_1, \dots, k_d) \in \mathbb{Z}^d$ and $z \in \mathbb{Z}$ such that $Q = 2^z((0,1)^d + \bar{k})$. Let $Q_0 \in \mathcal{X}^d$ and let $\tau : \mathbb{R}^d \to \mathbb{R}^d$ be the affine mapping $\tau(x) = rx + x_0$, $r > 0$, $x_0 \in \mathbb{R}^d$ that maps Q_0 onto the unit cube $(0,1)^d$. We say that $Q \in \mathcal{Q}^d$ is Q_0-dyadic, if $\tau(Q)$ is dyadic. For $q \geqslant 1$ we define the *Q-dyadic maximal function* $M_q^{\Delta,Q}$ by

$$(M_q^{\Delta,Q} f)(x) := \sup_{\substack{Q' \in \mathcal{X}^d \\ \text{with } x \in Q' \\ \text{and } Q' \text{ is } Q\text{-dyadic}}} M_{Q',q} f.$$

In the special case $q = 1$ we define $M^{\Delta,Q} f := M_1^{\Delta,Q} f$. Moreover, the $(0,1)^d$-dyadic maximal functions will simply called M_q^{Δ} and M^{Δ}.

Note that $Mf \sim M^{\Delta,Q} f$ uniformly in $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Moreover, $M^{\Delta,Q}$ has the same properties as the usual dyadic maximal function. Let $\Omega \subset \mathbb{R}^d$ be an open set. Then $Q_1 \subset \Omega$ is called a maximal Q-dyadic cube of Ω if and only if Q_1 is Q-dyadic and there exists no Q-dyadic cube Q_2 with $Q_1 \subsetneqq Q_2 \subset \Omega$. If $Q = (0,1)^d$ we just speak of a maximal dyadic cube of Ω. Note that every maximal Q-dyadic cube Q_1 of the set $M^{\Delta,Q} f > \lambda$, with $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $\lambda > 0$, satisfies $M_{Q_1} f \sim \lambda$ uniformly in $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and Q_1.

Lemma 5.5. *Let φ be a proper N-function on $\mathbb{R}^d$ of class $\mathcal{A}_\infty$. Then there exists $\delta > 0$ and $A \geqslant 1$ such that for all $\mathcal{Q} \in \mathcal{Y}^d$, all $\{t_Q\}_{Q \in \mathcal{Q}}$, $t_Q \geqslant 0$, and all $f \in L^1_{\mathrm{loc}}$ with $M_Q f \neq 0$, $Q \in \mathcal{Q}$, holds*

$$\left\| \sum_{Q \in \mathcal{Q}} t_Q \left| \frac{f}{M_Q f} \right|^\delta \chi_Q \right\|_\varphi \leqslant A \left\| \sum_{Q \in \mathcal{Q}} t_Q \chi_Q \right\|_\varphi. \tag{5.5}$$

Proof. Let $\mathcal{Q} \in \mathcal{Y}^d$, $\{t_Q\}_{Q \in \mathcal{Q}}$ with $t_Q \geqslant 0$, and $f \in L^\varphi_{\mathrm{loc}}$. We will fix $\delta > 0$ and $A \geqslant 1$ later. For all $Q \in \mathcal{Q}$ we define $f_Q \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ by

$$f_Q := f \chi_Q.$$

Since Q is Q-dyadic, f_Q is zero outside of Q, and $M_Q f \neq 0$ we obtain

$$\left\{ M^{\Delta,Q} f_Q > \frac{2}{3} M_Q f \right\} = Q. \tag{5.6}$$

Let

$$E_Q^k = \left\{ x \in \mathbb{R}^d \colon M^{\Delta,Q} f_Q(x) > \frac{2}{3} 2^{(d+1)k} M_Q f \right\}, \tag{5.7}$$

where $k \in N_0$. By definition of E_Q^k and by (5.6) holds

$$E_Q^{k+1} \subset E_Q^k \subset \cdots \subset E_Q^0 = Q. \tag{5.8}$$

Claim 1. *For every maximal Q-dyadic cube V of E_Q^{k-1}*

$$|E_Q^k \cap V| \leqslant \frac{1}{2}|V|. \tag{5.9}$$

Proof of Claim 1. Let V be a maximal Q-dyadic cube of E_Q^{k-1} and let W be a maximal Q-dyadic cubes of E_Q^k that intersects V. Since $E_Q^k \subset E_Q^{k-1}$, there holds $W \subset V$. Thus $W \subset E_Q^k \cap V$. Since W is maximal Q-dyadic in E_Q^k there holds (special property of the *dyadic* maximal function) $M_W f_Q > \frac{2}{3}2^{(d+1)k} M_Q f$. This implies

$$|W| M_Q f \leqslant \frac{3}{2} 2^{-(d+1)k} \int_W |f_Q| \,\mathrm{d}x.$$

Summing over all possible maximal Q-dyadic cubes W of E_Q^k that intersect V implies

$$|E_Q^k \cap V| M_Q f \leqslant \frac{3}{2} 2^{-(d+1)k} \int_V |f_Q| \,\mathrm{d}x. \tag{5.10}$$

Since V is maximal Q-dyadic in E^{k-1}, holds $M_{2V} f_Q \leqslant \frac{2}{3} 2^{(d+1)(k-1)} M_Q f$. Thus

$$\int_V |f_Q| \,\mathrm{d}x \leqslant 2^d |V| M_{2V} f_Q \leqslant \frac{2}{3} 2^d 2^{(d+1)(k-1)} |V| M_Q f. \tag{5.11}$$

Now (5.10), (5.11), and $M_Q f \neq 0$ imply

$$|E_Q^k \cap V| \leqslant \frac{1}{2}|V|.$$

This proves the claim. □

Let $\{V_{Q,l}^{k-1}\}_l$ be the collection of all maximal Q-dyadic cubes of E_Q^{k-1}, then

$$|E_Q^k \cap V_{Q,l}^{k-1}| \leqslant \frac{1}{2} |V_{Q,l}^{k-1}|. \tag{5.12}$$

Since $E_Q^{k-1} \subset Q$ and the family $\mathcal{Q}$ is pairwise disjoint, it follows that the collection $\{V_{Q,l}^{k-1}\}_{Q,l}$ is pairwise disjoint with respect to Q, l. Let

$$G^k := \bigcup_{Q \in \mathcal{Q}} E_Q^k, \qquad \Omega^k := \bigcup_{Q,l} V_{Q,l}^{k-1}.$$

Then

$$|G^k \cap V_{Q,l}^{k-1}| = |E_Q^k \cap V_{Q,l}^{k-1}| \leqslant \frac{1}{2} |V_{Q,l}^{k-1}|,$$

Thus we can apply Lemma 5.3 to G^k and Ω^k to get

$$\Big\| \sum_{Q \in \mathcal{Q}} \sum_l t_Q \chi_{G^k \cap V_{Q,l}^{k-1}} \Big\|_\varphi \leqslant \beta \Big\| \sum_{Q \in \mathcal{Q}} \sum_l t_Q \chi_{V_{Q,l}^{k-1}} \Big\|_\varphi$$

for some $0 < \beta < 1$ independent of $\mathcal{Q}$ and $\{t_Q\}_{Q\in\mathcal{Q}}$. Since $\bigcup_l V_{Q,l}^{k-1} = E_Q^{k-1}$, there follows

$$\Big\|\sum_{Q\in\mathcal{Q}} t_Q \chi_{G^k\cap E_Q^{k-1}}\Big\|_\varphi \leqslant \beta \Big\|\sum_{Q\in\mathcal{Q}} t_Q\chi_{E_Q^{k-1}}\Big\|_\varphi.$$

The definition of G^k and (5.8) imply $G^k\cap E_Q^{k-1} = E_Q^k \cap E_Q^{k-1} = E_Q^k$. Thus,

$$\Big\|\sum_{Q\in\mathcal{Q}} t_Q \chi_{E_Q^k}\Big\|_\varphi \leqslant \beta \Big\|\sum_{Q\in\mathcal{Q}} t_Q\chi_{E_Q^{k-1}}\Big\|_\varphi.$$

By induction

$$\Big\|\sum_{Q\in\mathcal{Q}} t_Q \chi_{E_Q^k}\Big\|_\varphi \leqslant \beta^k \Big\|\sum_{Q\in\mathcal{Q}} t_Q\chi_{E_Q^0}\Big\|_\varphi = \beta^k \Big\|\sum_{Q\in\mathcal{Q}} t_Q\chi_Q\Big\|_\varphi. \tag{5.13}$$

There follows

$$\begin{aligned}
\Big\|\sum_{Q\in\mathcal{Q}} t_Q \Big|\frac{f_Q}{M_Q f}\Big|^\delta \chi_{E_Q^k\setminus E_Q^{k+1}}\Big\|_\varphi &\leqslant \Big\|\sum_{Q\in\mathcal{Q}} t_Q \Big(\frac{M^{\Delta,Q} f_Q}{M_Q f}\Big)^\delta \chi_{E_Q^k\setminus E_Q^{k+1}}\Big\|_\varphi \\
&\leqslant \Big\|\sum_{Q\in\mathcal{Q}} t_Q \Big(\frac{2}{3}2^{(d+1)(k+1)}\Big)^\delta \chi_{E_Q^k\setminus E_Q^{k+1}}\Big\|_\varphi \quad \text{by (5.7)}\\
&\leqslant 2^{(d+1)(k+1)\delta}\Big\|\sum_{Q\in\mathcal{Q}} t_Q\chi_{E_Q^k}\Big\|_\varphi \\
&\leqslant 2^{(d+1)(k+1)\delta}\beta^k\Big\|\sum_{Q\in\mathcal{Q}} t_Q\chi_Q\Big\|_\varphi \quad \text{by (5.13) for } E_Q^k.
\end{aligned}$$

We fix $\delta > 0$ such that $\varepsilon := 2^{(d+1)\delta}\beta < 1$ and $(d+1)\delta \leqslant 1$. Then

$$\Big\|\sum_{Q\in\mathcal{Q}} t_Q \Big|\frac{f_Q}{M_Q f}\Big|^\delta \chi_{E_Q^k\setminus E_Q^{k+1}}\Big\|_\varphi \leqslant 2\varepsilon^k \Big\|\sum_{Q\in\mathcal{Q}} t_Q\chi_Q\Big\|_\varphi. \tag{5.14}$$

This implies

$$\begin{aligned}
\Big\|\sum_{Q\in\mathcal{Q}} t_Q \Big|\frac{f}{M_Q f}\Big|^\delta \chi_Q\Big\|_\varphi &\overset{(5.8)}{=} \Big\|\sum_{Q\in\mathcal{Q}} t_Q \Big|\frac{f_Q}{M_Q f}\Big|^\delta \chi_{E_Q^0}\Big\|_\varphi \\
&= \Big\|\sum_{k=0}^\infty\sum_{Q\in\mathcal{Q}} t_Q \Big|\frac{f_Q}{M_Q f}\Big|^\delta \chi_{E_Q^k\setminus E_Q^{k+1}}\Big\|_\varphi \\
&\leqslant \sum_{k=0}^\infty \Big\|\sum_{Q\in\mathcal{Q}} t_Q \Big|\frac{f_Q}{M_Q f}\Big|^\delta \chi_{E_Q^k\setminus E_Q^{k+1}}\Big\|_\varphi \\
&\overset{(5.14)}{\leqslant} \sum_{k=0}^\infty 2\varepsilon^k \Big\|\sum_{Q\in\mathcal{Q}} t_Q\chi_Q\Big\|_\varphi
\end{aligned}$$

$$= \frac{2}{1-\varepsilon}\left\| \sum_{Q\in\mathcal{Q}} t_Q \chi_Q \right\|_\varphi.$$

This proves the lemma. □

Theorem 5.6. *Let φ be a proper N-function on $\mathbb{R}^d$ of class $\mathcal{A}_\infty$. Then there exists $s > 1$, such that $(M_{s,Q}\varphi)(t) \lesssim (M_Q\varphi)(t)$.*

Proof. Let φ be as required. Due to Lemma 5.5 there exists $\delta > 0$ and $A \geqslant 1$ such that for all $\mathcal{Q} \in \mathcal{Y}^d$, all $\{t_Q\}_{Q\in\mathcal{Q}}$ with $t_Q \geqslant 0$, and all $f \in L^1_{\mathrm{loc}}$ with $M_Q f \neq 0$, $Q \in \mathcal{Q}$, holds

$$\left\| \sum_{Q\in\mathcal{Q}} t_Q \left|\frac{f}{M_Q f}\right|^\delta \chi_Q \right\|_\varphi \leqslant A \left\| \sum_{Q\in\mathcal{Q}} t_Q \chi_Q \right\|_\varphi. \tag{5.15}$$

Define $s := 1 + \delta$. Let $\mathcal{Q} \in \mathcal{Y}^d$ and $\{u_Q\}_{Q\in\mathcal{Q}}$ with $u_Q > 0$ be such that

$$\sum_{Q\in\mathcal{Q}} |Q|(M_Q\varphi)(u_Q) \leqslant 1. \tag{5.16}$$

We have to show that

$$\sum_{Q\in\mathcal{Q}} |Q|(M_{s,Q}\varphi)(u_Q) \leqslant A_2, \tag{5.17}$$

where $A_2 \geqslant 1$ does not depend on $\mathcal{Q}$ nor $\{u_Q\}_{Q\in\mathcal{Q}}$. Due to (5.16) there holds

$$\int_{\mathbb{R}^d} \varphi\left(\sum_{Q\in\mathcal{Q}} \chi_Q u_Q \right) \mathrm{d}x \leqslant 1. \tag{5.18}$$

This implies

$$\left\| \sum_{Q\in\mathcal{Q}} \chi_Q u_Q \right\|_\varphi \leqslant 1. \tag{5.19}$$

Define $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$

$$f := \sum_{Q\in\mathcal{Q}} \chi_Q \varphi(u_Q),$$

then $M_Q f \neq 0$ for all $Q \in \mathcal{Q}$. Now (5.15) implies

$$\left\| \sum_{Q\in\mathcal{Q}} u_Q \left|\frac{f}{M_Q f}\right|^\delta \chi_Q \right\|_\varphi \leqslant A.$$

This and the convexity of φ implies

$$C \geqslant \sum_{Q\in\mathcal{Q}} \int_Q \varphi\left(u_Q \left|\frac{f}{M_Q f}\right|^\delta \right) \mathrm{d}x \geqslant \sum_{Q\in\mathcal{Q}} \int_Q \varphi(u_Q) \left|\frac{f}{M_Q f}\right|^\delta \chi_{|f|\geqslant M_Q f}\, \mathrm{d}x. \tag{5.20}$$

On the other hand (5.18) implies

$$1 \geqslant \sum_{Q \in \mathcal{Q}} \int_Q \varphi(u_Q)\,\mathrm{d}x \geqslant \sum_{Q \in \mathcal{Q}} \int_Q \varphi(u_Q) \left| \frac{f}{M_Q f} \right|^{\delta} \chi_{|f| < M_Q f}\,\mathrm{d}x. \tag{5.21}$$

Overall, (5.20), (5.21), and $s = 1 + \delta$ imply

$$\begin{aligned} C &\geqslant \sum_{Q \in \mathcal{Q}} \int_Q \varphi(u_Q) \left| \frac{f}{M_Q f} \right|^{\delta} \mathrm{d}x \\ &= \sum_{Q \in \mathcal{Q}} \int_Q \big(\varphi(u_Q)\big)^{1+\delta}\,\mathrm{d}x \big((M_Q\varphi)(u_Q)\big)^{-\delta} \\ &= \sum_{Q \in \mathcal{Q}} |Q| \big((M_{1,s}\varphi)(u_Q)\big)^{1+\delta} \big((M_Q\varphi)(u_Q)\big)^{-\delta} \\ &\geqslant \sum_{Q \in \mathcal{Q}} |Q| (M_{1,s}\varphi)(u_Q). \end{aligned} \tag{5.22}$$

This proves (5.17), which concludes the proof. □

Theorem 5.6 provides a kind of reverse Hölder estimates. We remark that in the case of weighted (classical) Lebesgue spaces, i.e. $\varphi(x,t) = t^q \omega(x)$, this matches exactly the reverse Hölder estimate for Muckenhoupt weights $\omega \in A_q$. Let us summarize our results so far.

Theorem 5.7. *Let φ be a proper N-function on $\mathbb{R}^d$. Then the following conditions are equivalent*:

(a) *φ is of class $\mathcal{A}$.*
(b) *$M_Q\varphi \lesssim (M_Q\varphi^*)^*$.*
(c) *There exists $s > 1$ such that $M_{s,Q}\varphi \lesssim M_Q\varphi \lesssim (M_Q\varphi^*)^* \lesssim (M_{s,Q}\varphi^*)^*$.*

Proof. Let φ be a proper N-function on $\mathbb{R}^d$. Then (a) $\Leftrightarrow$ (b) follows from Theorem 3.6 while (c) $\Rightarrow$ (b) is obvious. We will show (a, b) $\Rightarrow$ (c), so let φ be of class $\mathcal{A}$ or equivalently $M_Q\varphi \lesssim (M_Q\varphi^*)^*$. Then by Lemma 3.3 φ and φ^* are of class $\mathcal{A}$, so by Lemma 5.2 φ and φ^* are of class $\mathcal{A}_\infty$. Hence, by Theorem 5.6 there exists $s > 1$ such that $M_{s,Q}\varphi \lesssim M_Q\varphi$ and $M_{s,Q}\varphi^* \lesssim M_Q\varphi^*$. From (2.6) follows $(M_Q\varphi^*)^* \lesssim (M_{s,Q}\varphi^*)^*$. We obtain $M_{s,Q}\varphi \lesssim M_Q\varphi \lesssim (M_Q\varphi^*)^* \lesssim (M_{s,Q}\varphi^*)^*$. This proves the claim. □

To get a better understanding of Theorem 5.7 we need to examine the N-functions $M_{s,Q}\varphi$ and $(M_{s,Q}\varphi^*)^*$.

Lemma 5.8. *Let Ω is either be $\mathbb{R}^d$ or $\mathcal{X}^d$ and let $r, s > 0$. Let φ be a proper N-function on Ω and let $\gamma : \Omega \times \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}$ be defined by*

$$\gamma(t) := \int_0^t \left(\varphi'(u^{1/r})\right)^s \mathrm{d}u. \tag{5.23}$$

Then γ is a proper N-function on Ω with

$$\frac{\gamma(\omega, t^r)}{t^r} \sim \left(\frac{\varphi(\omega, t)}{t}\right)^s, \qquad \frac{\gamma^*(\omega, t^s)}{t^s} \sim \left(\frac{\varphi^*(\omega, t)}{t}\right)^r \tag{5.24}$$

uniformly in $\omega \in \Omega$ and $t > 0$. If ψ is another N-function on Ω such that

$$\frac{\psi(\omega, t^r)}{t^r} \sim \left(\frac{\varphi(\omega, t)}{t}\right)^s \tag{5.25}$$

uniformly in $\omega \in \Omega$ and $t > 0$, then ψ is a proper N-function on Ω and

$$\frac{\psi^*(\omega, t^s)}{t^s} \sim \left(\frac{\varphi^*(\omega, t)}{t}\right)^r \tag{5.26}$$

uniformly in $\omega \in \Omega$ and $t > 0$.

Proof. Since all following calculations are uniform with respect to ω, we will omit the dependence on ω. From the definition of γ it follows immediately that γ is an N-function on Ω and (2.3) implies

$$\frac{\gamma(t^r)}{t^r} \sim \gamma'(t^r) = \left(\varphi'(t)\right)^s \sim \left(\frac{\varphi(t)}{t}\right)^s. \tag{5.27}$$

From $\gamma'(t^r) = ((\varphi')(t))^s$ we deduce $((\varphi')^{-1}(t))^r = (\gamma')^{-1}(t^s)$. Thus

$$\left((\varphi^*)'(t)\right)^r = \left((\varphi')^{-1}(t)\right)^r = (\gamma')^{-1}(t^s) = (\gamma^*)'(t^s).$$

Hence (2.3) implies

$$\left(\frac{\varphi^*(t)}{t}\right)^r \sim \left((\varphi^*)'(t)\right)^r = (\gamma^*)'(t^s) \sim \frac{\gamma^*(t^s)}{t^s}. \tag{5.28}$$

Since φ and φ^* satisfy the strong Δ_2-condition, we immediately deduce from (5.27) and (5.28) that γ and γ^* satisfy the strong Δ_2-condition. From (5.25) and (5.27) we deduce that $\psi \sim \gamma$. Thus there exists $c_0, c_1 > 0$ with

$$c_0\gamma(t) \leqslant \psi(t) \leqslant c_1\gamma(t).$$

Thus by (2.5) and (2.6)

$$c_1\gamma^*\left(\frac{t}{c_1}\right) \leqslant \psi^*(t) \leqslant c_0\gamma^*\left(\frac{t}{c_0}\right).$$

Since γ^* satisfies the strong Δ_2-condition this implies $\gamma^* \sim \psi^*$. Overall, we have shown $\gamma \sim \psi$ and $\gamma^* \sim \psi^*$. So (5.26) and the strong Δ_2-condition follow from the properties of γ. This proves the lemma. $\square$

6. Sufficient condition

If φ is a proper N-function on $\mathbb{R}^d$ and M is continuous on $L^\varphi(\mathbb{R}^d)$ then we deduce from Sections 3 and 5 that φ is of class $\mathcal{A}$ and $M_Q\varphi \lesssim (M_{s,Q}\varphi^*)^*$ for some $s > 1$. Thus, $M_Q\varphi \lesssim (M_{s,Q}\varphi^*)^*$ for some $s > 1$ is a necessary condition for M to be continuous on $L^\varphi(\mathbb{R}^d)$. In the following we will define a new relation $\ll$ which is slightly stronger than $\lesssim$. We will show that $M_Q\varphi \ll (M_{s,Q}\varphi^*)^*$ for some $s > 1$ is sufficient for the continuity of M and, even more, M_q for some $q > 1$ on $L^{p(\cdot)}(\mathbb{R}^d)$.

Definition 6.1. Let $\varphi, \psi : \mathcal{X}^d \times \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}$. We say that ψ is *strongly dominated* by φ or shortly $\psi \ll \varphi$ if $\psi \lesssim \varphi$ and for all $A_1 > 0$ there exist $A_2 > 0$ such that the following holds:

For all families $\mathcal{Q}_\lambda \in \mathcal{Y}^d$, $\lambda > 0$, with

$$\sum_{Q \in \mathcal{Q}_\lambda} |Q|\varphi(Q,\lambda) \leqslant A_1 \tag{6.1}$$

and

$$\int_0^\infty \lambda^{-1} \sum_{Q \in \mathcal{Q}_\lambda} |Q|\varphi(Q,\lambda)\,\mathrm{d}\lambda \leqslant A_1, \tag{6.2}$$

there holds

$$\int_0^\infty \lambda^{-1} \sum_{Q \in \mathcal{Q}_\lambda} |Q|\psi(Q,\lambda)\,\mathrm{d}\lambda \leqslant A_2. \tag{6.3}$$

Note that due to the strong Δ_2-condition is suffices to verify (3.1) for one couple $A_1, A_2 > 0$. The purpose of the new relation $\ll$ is the following: If $\psi \lesssim \varphi$ then the integrand in (6.3) is bounded by $C\lambda^{-1}$ for some $C > 0$. But this does not ensure the boundedness of the integral, while $\psi \ll \varphi$ does. See Section 7 for more details on the difference between $\psi \lesssim \varphi$ and $\psi \ll \varphi$. Nevertheless we will see later in Section 8 that in the case of generalized Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^d)$, i.e. $\varphi(x,t) = t^{p(x)}$, we can pass from domination to strong domination: We will see that $M_Q\varphi \lesssim (M_Q\varphi^*)^*$ implies that $M_{s,Q}\varphi \ll (M_{s,Q}\varphi^*)^*$ for some $s > 1$, especially $M_Q\varphi \ll (M_{s,Q}\varphi^*)^*$. We will now show that this implies the continuity of M^Δ and M_q^Δ for some $q > 1$ on $L^\varphi(\mathbb{R}^d)$. Then later in Theorem 6.4 we will show that it even implies the continuity of M and M_q for some $q > 1$ on $L^\varphi(\mathbb{R}^d)$.

Theorem 6.2. *Let φ be a proper N-function on $\mathbb{R}^d$ and let $s > 1$ be such that $M_Q\varphi \ll (M_{s,Q}\varphi^*)^*$. Then there exists $q > 1$ such that M_q^Δ is continuous on $L^\varphi(\mathbb{R}^d)$. Note, that M_q^Δ is the dyadic maximal function.*

Proof. Since φ^* satisfies the strong Δ_2-condition it follows from [17, Lemma 1.2.2+1.2.3], that $\varphi^{1/(1+\varepsilon)}$ is quasiconvex for some $\varepsilon > 0$. As a consequence there exists a proper N-function ρ with $\varphi \sim \rho^{1+\varepsilon}$. Hence, for $t \geqslant 0$ and $u \geqslant 1$

$$\varphi(x, ut) \geqslant c\rho^{1+\varepsilon}(x, ut) \geqslant cu^{1+\varepsilon}\rho^{1+\varepsilon}(x, t) \geqslant cu^{1+\varepsilon}\varphi(x, t).$$

Overall, we have shown that there exists $C_4 \geqslant 1$ and $\varepsilon > 0$ such that for all $x \in \mathbb{R}^d$, $t \geqslant 0$, and $u \geqslant 1$

$$\varphi(x, ut) \geqslant C_4 u^{1+\varepsilon}\varphi(x, t). \tag{6.4}$$

Let $r_1 \in \mathbb{R}$ such that $1 < r_1 \leqslant s$. Then there exists $q > 1$ such that

$$qr_1 \leqslant s, \quad \frac{1}{q} > 1 - \varepsilon(r_1 - 1). \tag{6.5}$$

Now let $r_0 \in \mathbb{R}$ such that $0 < r_0 < 1$ and $qr_0 \geqslant 1$. Then for $j = 0, 1$

$$1 \leqslant qr_j \leqslant s, \quad \varepsilon + \frac{\frac{1}{q} - 1}{r_j - 1} > 0. \tag{6.6}$$

Since $f \mapsto M_q^\Delta f$ is sub-linear it suffices to show that $f \mapsto M_q^\Delta f$ is bounded. Thus it suffices to show that there exists $A > 0$ such that for all $f \in L^\varphi(\mathbb{R}^d)$

$$\int \varphi(f)\,\mathrm{d}x \leqslant 1 \quad \Rightarrow \quad \int \varphi(M_q^\Delta f)\,\mathrm{d}x \leqslant A. \tag{6.7}$$

Let $f \in L^\varphi(\mathbb{R}^d)$ with $\int \varphi(f)\,\mathrm{d}x \leqslant 1$. For $\lambda > 0$ define $f_{0,\lambda}, f_{1,\lambda} : \mathbb{R}^d \to \mathbb{R}$ by

$$f_{0,\lambda} := f\chi_{\{|f| \leqslant \lambda\}},$$
$$f_{1,\lambda} := f\chi_{\{|f| > \lambda\}}.$$

Then

$$\{M_q^\Delta f > \lambda\} \subset \{M_q^\Delta f_{0,\lambda} > \lambda/2\} \cup \{M_q^\Delta f_{1,\lambda} > \lambda/2\}. \tag{6.8}$$

This implies

$$\begin{aligned}\int_{\mathbb{R}^d} \varphi(M_q^\Delta f)\,\mathrm{d}x &= \int_0^\infty \int_{\mathbb{R}^d} \varphi'(\lambda)\chi_{\{M_q^\Delta f > \lambda\}}\,\mathrm{d}x\,\mathrm{d}\lambda \\ &\leqslant C\sum_{j=0}^1 \int_0^\infty \lambda^{-1} \int_{\mathbb{R}^d} \varphi(\lambda)\chi_{\{M_q^\Delta f_{j,\lambda} > \lambda/2\}}\,\mathrm{d}x\,\mathrm{d}\lambda \quad \text{by (2.3) and (6.8).}\end{aligned}$$

For $\lambda > 0$ and $j = 0, 1$ let $\mathcal{Q}_{j,\lambda}$ be the decomposition of $\{M_q^\Delta f_{j,\lambda} > \lambda/2\}$ into maximal dyadic cubes. Then for all $Q \in \mathcal{Q}_{j,\lambda}$ there holds (uniformly in Q)

$$M_{q,Q} f_{j,\lambda} \sim \lambda. \tag{6.9}$$

Moreover,

$$\int_{\mathbb{R}^d} \varphi(M_q^\Delta f)\,\mathrm{d}x \leqslant C\sum_{j=0}^1 \int_0^\infty \lambda^{-1} \sum_{Q \in \mathcal{Q}_{j,\lambda}} |Q|(M_Q\varphi)(\lambda)\,\mathrm{d}\lambda. \tag{6.10}$$

From $1 \leqslant qr_0 < qr_1 \leqslant s$, Jensen's inequality, and (2.6) we deduce

$$(M_{s,Q}\varphi^*)^* \leqslant (M_{qr_j,Q}\varphi^*)^*,$$

for $j = 0, 1$. Thus $M_Q\varphi \ll (M_{s_1,Q}\varphi^*)^*$ implies $M_Q\varphi \ll (M_{qr_j,Q}\varphi^*)^*$, $j = 0, 1$. We will show that (uniformly in $\mathcal{Q} \in \mathcal{Y}^d$ and f with $\|f\|_\varphi \leqslant 1$)

$$\sum_{Q \in \mathcal{Q}_{j,\lambda}} |Q|(M_{qr_j,Q}\varphi^*)^*(\lambda) \leqslant C \quad \text{for } j = 0, 1, \tag{6.11}$$

uniformly in $\lambda > 0$ and that

$$\int_0^\infty \lambda^{-1} \sum_{Q \in \mathcal{Q}_{j,\lambda}} |Q|(M_{qr_j,Q}\varphi^*)^*(\lambda)\, \mathrm{d}\lambda \leqslant C \quad \text{for } j = 0, 1, \tag{6.12}$$

since then the strong dominations $M_Q\varphi \ll (M_{qr_j,Q}\varphi^*)^*$, $j = 0, 1$ imply the boundedness of the right-hand side of (6.10). This proves (6.7) which concludes the proof of the theorem. It remains to prove (6.11) and (6.12).

We point out once more that the constant C may change from line to line but will not depend on f, λ, nor $Q \in \mathcal{Q}_{j,\lambda}$. Define $\psi_0, \psi_1 : \mathbb{R}^d \times \mathbb{R}^{\geqslant} \to \mathbb{R}^{\geqslant 0}$ by

$$\psi_j(t) := \int_0^t (\varphi)'(u^{1/r_j})^{1/q}\, \mathrm{d}u, \qquad j = 0, 1 \tag{6.13}$$

then by Lemma 5.8 ψ is a proper N-function on $\mathbb{R}^d$ and

$$\left(\frac{\varphi(x,t)}{t}\right)^{1/q} \sim \frac{\psi_j(x, t^{r_j})}{t^{r_j}}, \tag{6.14}$$

$$\left(\frac{M_{qr_j,Q}\varphi^*(t)}{t}\right)^{r_j} \sim \frac{M_{q,Q}\psi_j^*(t^{1/q})}{t^{1/q}}. \tag{6.15}$$

Since $qr_j \geqslant 1$ and $q > 1$, $M_{qr_j,Q}\varphi^*$ and $M_q\psi_j^*$ are proper N-function by Lemma 3.4. Thus Lemma 5.8 implies

$$\left(\frac{(M_{qr_j,Q}\varphi^*)^*(t)}{t}\right)^{1/q} \sim \frac{(M_{q,Q}\psi_j^*)^*(t^{r_j})}{t^{r_j}}. \tag{6.16}$$

From $\int_{\mathbb{R}^d} \varphi(|f|)\, \mathrm{d}x \leqslant 1$ we deduce

$$\sum_{Q \in \mathcal{Q}_{j,\lambda}} |Q| M_Q(\varphi(|f_{j,\lambda}|)) \leqslant \int_{\mathbb{R}^d} \varphi(|f|)\, \mathrm{d}x \leqslant 1. \tag{6.17}$$

By definition of $f_{j,\lambda}$ and r_j it holds for $j = 0, 1$ and $\lambda > 0$

$$\left(\frac{|f_{j,\lambda}|}{\lambda}\right)^{r_j - 1} \geqslant \chi_{\{f_{j,\lambda} \neq 0\}}. \tag{6.18}$$

This, (6.4), and (6.6) implies for $j = 0, 1$

$$
\begin{aligned}
&\psi_j\big(|f_{j,\lambda}|\lambda^{r_j-1}\big)\lambda^{-r_j+1/q}\\
&\quad\leqslant C\psi_j\big(|f_{j,\lambda}|^{r_j}\big)\left(\frac{|f_{j,\lambda}|}{\lambda}\right)^{-(r_j-1)(1+\varepsilon)}\lambda^{-r_j+1/q}\chi_{\{f_{j,\lambda}\neq 0\}}\quad\text{by (6.4), (6.18)}\\
&\quad= C\psi_j\big(|f_{j,\lambda}|^{r_j}\big)|f_{j,\lambda}|^{-(r_j-1)(1+\varepsilon)}\lambda^{1/q-1+\varepsilon(r_j-1)}\chi_{\{f_{j,\lambda}\neq 0\}}\\
&\quad\leqslant C\psi_j\big(|f_{j,\lambda}|^{r_j}\big)|f_{j,\lambda}|^{-r_j+1/q}\chi_{\{f_{j,\lambda}\neq 0\}}\quad\text{by (6.6), (6.18)}\\
&\quad\leqslant C\big(\varphi\big(|f_{j,\lambda}|\big)\big)^{1/q}\quad\text{by (6.14).}
\end{aligned}
\tag{6.19}
$$

Thus we deduce

$$
\begin{aligned}
&(M_{qr_j,Q}\varphi^*)^*(\lambda)\\
&\quad\leqslant C\big((M_{q,Q}\psi_j^*)^*(\lambda^{r_j})\lambda^{-r_j+1/q}\big)^q\quad\text{by (6.16)} + \text{strong } \Delta_2\\
&\quad\leqslant C\big((M_{q,Q}\psi_j^*)^*\big((M_{q,Q}f_{j,\lambda})\lambda^{r_j-1}\big)\lambda^{-r_j+1/q}\big)^q\quad\text{by (6.9)}\\
&\quad\leqslant C\big(M_{q,Q}\big(\psi_j\big(|f_{j,\lambda}|\lambda^{r_j-1}\big)ig\big)\lambda^{-r_j+1/q}\big)^q\quad\text{by Lemma 3.7}\\
&\quad\leqslant C\big(M_{q,Q}\big(\psi_j\big(|f_{j,\lambda}|^{r_j}\big)|f_{j,\lambda}|^{-(r_j-1)(1+\varepsilon)}\lambda^{1/q-1+\varepsilon(r_j-1)}\chi_{\{f_{j,\lambda}\neq 0\}}\big)\big)^q\quad\text{by (6.19)}\\
&\quad\leqslant CM_Q\big(\varphi\big(|f_{j,\lambda}|\big)\big)\quad\text{by (6.19).}
\end{aligned}
\tag{6.20}
$$

Thus (6.17) and (6.20) prove (6.11). From (6.19) and (6.20) we further deduce

$$
\begin{aligned}
&\int_0^\infty \lambda^{-1}\sum_{Q\in\mathcal{Q}_{j,\lambda}}|Q|(M_{qr_j,Q}\varphi^*)^*(\lambda)\,\mathrm{d}\lambda\\
&\quad\leqslant C\int_0^\infty \lambda^{-1}\sum_{Q\in\mathcal{Q}_{j,\lambda}}|Q|\big(M_{q,Q}\big(\psi_j\big(|f_{j,\lambda}|^{r_j}\big)\\
&\qquad\times|f_{j,\lambda}|^{-(r_j-1)(1+\varepsilon)}\lambda^{\frac{1}{q}-1+\varepsilon(r_j-1)}\chi_{\{f_{j,\lambda}\neq 0\}}\big)\big)^q\,\mathrm{d}\lambda\\
&\quad\leqslant C\int_{\mathbb{R}^d}\int_0^\infty\big(\psi_j\big(|f_{j,\lambda}|^{r_j}\big)\big)^q|f_{j,\lambda}|^{-q(r_j-1)(1+\varepsilon)}\lambda^{q(-1+\varepsilon(r_j-1))}\chi_{\{f_{j,\lambda}\neq 0\}}\,\mathrm{d}\lambda\,\mathrm{d}x\\
&\quad=: I_j.
\end{aligned}
\tag{6.21}
$$

From $q>1$, $r_0<1$, and (6.5) we deduce

$$
q\big(-1+\varepsilon(r_0-1)\big)<-1,\qquad q\big(-1+\varepsilon(r_1-1)\big)>-1.
\tag{6.22}
$$

From the definition of $f_{j,\lambda}$ we deduce

$$
\begin{aligned}
I_0&=C\int_{\mathbb{R}^d}\int_{|f|}^\infty\big(\psi_0\big(|f|^{r_0}\big)\big)^q|f|^{-q(r_0-1)(1+\varepsilon)}\lambda^{q(-1+\varepsilon(r_0-1))}\,\mathrm{d}\lambda\,\mathrm{d}x\\
&=C(q,r_0,\varepsilon)\int_{\mathbb{R}^d}\big(\psi_0\big(|f|^{r_0}\big)\big)^q|f|^{-qr_0+1}\,\mathrm{d}x\quad\text{by (6.22)}
\end{aligned}
$$

$$\leqslant C(q,r_0,\varepsilon)\int\limits_{\mathbb{R}^d}\varphi\bigl(|f|\bigr)\,\mathrm{d}x \quad \text{by (6.14)}$$
$$\leqslant C(q,r_0,\varepsilon)$$

and analogously

$$I_1 = C\int\limits_{\mathbb{R}^d}\int\limits_0^{|f|}\bigl(\psi_1\bigl(|f|^{r_1}\bigr)\bigr)^q|f|^{-q(r_1-1)(1+\varepsilon)}\lambda^{q(-1+\varepsilon(r_1-1))}\,\mathrm{d}\lambda\,\mathrm{d}x$$
$$= C(q,r_1,\varepsilon)\int\limits_{\mathbb{R}^d}\bigl(\psi_1\bigl(|f|^{r_1}\bigr)\bigr)^q|f|^{-qr_1+1}\,\mathrm{d}x \quad \text{by (6.22)}$$
$$\leqslant C(q,r_1,\varepsilon)\int\limits_{\mathbb{R}^d}\varphi\bigl(|f|\bigr)\,\mathrm{d}x \quad \text{by (6.14)}$$
$$\leqslant C(q,r_1,\varepsilon).$$

Overall,

$$\int\limits_0^\infty \lambda^{-1}\sum_{Q\in\mathcal{Q}_{j,\lambda}}(M_{qr_j,Q}\varphi^*)^*(\lambda)\,\mathrm{d}\lambda \leqslant I_j \leqslant C(q,r_0,r_1,\varepsilon),$$

for $j=0,1$. This proves (6.12). The theorem is proven.

Lemma 6.3. *Let φ be a proper N-function on $\mathbb{R}^d$, let $M_Q\varphi \ll (M_Q\varphi^*)^*$, and let M_q^Δ be continuous on $L^\varphi(\mathbb{R}^d)$ for some $q\geqslant 1$. Then M_q is continuous on $L^\varphi(\mathbb{R}^d)$.*

Proof. The proof is closely related to distributional inequalities and good-λ-estimates. This becomes clear if we explain the case $\varphi(x,t)=\varphi(t)$ first: From [31, p. 188] we know that $|\{M_q f > c_0\lambda\}| \leqslant c|\{M_q^\Delta f > \lambda\}|$ for suitable $c_0>0$. Now, $\int_{\mathbb{R}^d}\varphi(|M_q f(x)|)\,\mathrm{d}x = \int\varphi'(\lambda)|M_q f>\lambda|\,\mathrm{d}\lambda$ and the analogue for M_q^Δ shows

$$\int\limits_{\mathbb{R}^d}\varphi\bigl(\bigl|M_q f(x)\bigr|\bigr)\,\mathrm{d}x \leqslant c\int\limits_{\mathbb{R}^d}\varphi\bigl(\bigl|M_q^\Delta f(x)\bigr|\bigr)\,\mathrm{d}x.$$

Especially, for all f with $\|M_q^\Delta f\|_\varphi \leqslant 1$ follows $\|M_q f\|_\varphi \leqslant c$ which proves the lemma in the case $\varphi(x,t)=\varphi(t)$.

We will now study the general case, where φ may depend on x. Let $f\in L^\varphi$ with $\|M_q^\Delta f\|_\varphi\leqslant 1$, then $\int\varphi(x,M_q^\Delta f(x))\,\mathrm{d}x\leqslant 1$. It suffices to show $\int\varphi(x,M_q f(x))\,\mathrm{d}x\leqslant C$. We estimate with (2.3)

$$\int\varphi(M_q f)\,\mathrm{d}x \leqslant c\int\limits_0^\infty\lambda^{-1}\int\limits_{\mathbb{R}^d}\chi_{\{M_q f>\lambda\}}\varphi(x,\lambda)\,\mathrm{d}x\,\mathrm{d}\lambda. \tag{6.23}$$

For $\lambda>0$ fixed let $\mathcal{Q}_\lambda\in\mathcal{Y}^d$ be the decomposition of $\{M_q^\Delta f>\lambda\}$ into maximal dyadic cubes. Then as in [31, p. 188] we can choose c_0 (only depending on the dimension d) such that $M_q f > c_0\,\lambda \subset \bigcup_{Q\in\mathcal{Q}} 2Q =: \Omega_\lambda$, where $2Q$ is the cube with the same center as Q but twice the diameter. Hence, by (6.23)

$$\int \varphi(M_q f)\,\mathrm{d}x \leqslant c\int_0^\infty \lambda^{-1}\int_{\Omega_\lambda}\varphi(x,\lambda)\,\mathrm{d}x\,\mathrm{d}\lambda. \tag{6.24}$$

Let $K_\lambda \Subset \Omega_\lambda$ be compact. For every $Q\in\mathcal{Q}_\lambda$ and every $x\in 2Q$ let W_x be the smallest cube which is centered at x but still contains Q. Then $Q\subset W_x\subset 4Q$, where $4Q$ is the cube with the same center as Q but four times the diameter. Especially, $|W_x|\sim|Q|$ for all $x\in 2Q$. Then we deduce with the help of (3.2), $|W_x|\sim|Q|$, the properness of $(M_Q\varphi^*)^*$, and $Q\subset W_x$ that

$$\begin{aligned}(M_{W_x}\varphi^*)^*(\lambda) &\leqslant (M_{W_x}\varphi^*)^*\bigl(cM_{W_x}(\lambda\,\chi_Q)\bigr) \overset{(3.2)}{\leqslant} c\frac{|Q|}{|W_x|}(M_Q\varphi)(\lambda)\\ &\leqslant c(M_Q\varphi)(\lambda).\end{aligned} \tag{6.25}$$

Let $\mathcal{W}_\lambda$ denote the collection of all W_x with $Q\in\mathcal{Q}_\lambda$ and $x\in 2Q$ then $\mathcal{W}_\lambda$ covers K_λ. Therefore, by Theorem 1.1 in [4] there exists a number N which only depends on the dimension d and there exist subcollections $\mathcal{W}_{\lambda,j}$, $j=1,\dots,N$, of $\mathcal{W}_\lambda$ of pairwise disjoint cubes which still cover K_λ. Especially, we have $\mathcal{W}_{\lambda,1},\dots,\mathcal{W}_{\lambda,N}\in\mathcal{Y}^d$ and $K_\lambda\subset\bigcup_{j=1}^N\bigcup_{Q\in\mathcal{W}_{\lambda,j}} Q$. This and (6.24) implies

$$\int_0^\infty \lambda^{-1}\int_{K_\lambda}\varphi(x,\lambda)\,\mathrm{d}x\,\mathrm{d}\lambda \leqslant c\int_0^\infty\lambda^{-1}\sum_{j=1}^N\sum_{W\in\mathcal{W}_{\lambda,j}}|W|(M_W\varphi)(\lambda)\,\mathrm{d}\lambda. \tag{6.26}$$

We want to show that the right-hand side of (6.26) is bounded by some constant. Since $M_Q\varphi \ll (M_Q\varphi^*)^*$ it suffices to show the following estimates

$$\sum_{j=1}^N\sum_{W\in\mathcal{W}_{\lambda,j}}|W|(M_W\varphi^*)^*(\lambda)\leqslant c, \tag{6.27}$$

$$\int_0^\infty\lambda^{-1}\sum_{j=1}^N\sum_{W\in\mathcal{W}_{\lambda,j}}|W|(M_W\varphi^*)^*(\lambda)\,\mathrm{d}\lambda\leqslant c. \tag{6.28}$$

Due to (6.25) and the construction of the collections $\mathcal{W}_{\lambda,j}$ there exists for every $W\in\mathcal{W}_{\lambda,j}$ a cube $Q\in\mathcal{Q}_\lambda$ with $Q\subset W\subset 4Q$

$$(M_W\varphi^*)^*(\lambda)\leqslant c(M_Q\varphi)(\lambda).$$

Therefore, instead of (6.27) and (6.28) it suffices to prove

$$\sum_{Q\in\mathcal{Q}_\lambda}|Q|(M_Q\varphi)(\lambda)\leqslant c, \tag{6.29}$$

$$\int_0^\infty\lambda^{-1}\sum_{Q\in\mathcal{Q}_\lambda}|Q|(M_Q\varphi)(\lambda)\,\mathrm{d}\lambda\leqslant c. \tag{6.30}$$

Since $\int \varphi(x, M_q f(x))\,\mathrm{d}x \leqslant 1$ we have with the help of (2.3)

$$\sum_{Q \in \mathcal{Q}_\lambda} |Q|(M_Q\varphi)(\lambda) = \int\limits_{M_q^\Delta f > \lambda} \varphi(x,\lambda)\,\mathrm{d}x \leqslant \int\limits_{\mathbb{R}^d} \varphi\big(x, M_q^\Delta f(x)\big)\,\mathrm{d}x \leqslant c. \tag{6.31}$$

The same arguments imply

$$\int\limits_0^\infty \lambda^{-1} \sum_{Q \in \mathcal{Q}_\lambda} |Q|(M_Q\varphi)(\lambda)\,\mathrm{d}\lambda \leqslant c \int\limits_0^\infty \int\limits_{\mathbb{R}^d} \varphi'(x,\lambda)\chi_{\{M_q^\Delta f > \lambda\}}\,\mathrm{d}x\,\mathrm{d}\lambda \tag{6.32}$$

$$= \int\limits_{\mathbb{R}^d} \varphi\big(x, M_q^\Delta f(x)\big)\,\mathrm{d}x \leqslant c. \tag{6.33}$$

This proves (6.29) and (6.30). This proves that the right-hand side of (6.26) is bounded, i.e.

$$\int\limits_0^\infty \lambda^{-1} \int\limits_{K_\lambda} \varphi(x,\lambda)\,\mathrm{d}x\,\mathrm{d}\lambda \leqslant c$$

for all $K_\lambda \Subset \Omega_\lambda$, where c is independent of the choice of the K_λ. Exhausting the sets Ω_λ by K_λ we derive with (6.24) that $\int \varphi(M_q f)\,\mathrm{d}x$ is bounded. This proves the lemma. □

Theorem 6.4. *Let φ be a proper N-function on $\mathbb{R}^d$ and let $s > 1$ be such that $M_Q\varphi \ll (M_{s,Q}\varphi^*)^*$. Then there exists $q > 1$ such that M_q is continuous on $L^\varphi(\mathbb{R}^d)$.*

Proof. Due to Theorem 6.2 there exists $q > 1$ such that M_q^Δ is continuous on $L^\varphi(\mathbb{R}^d)$. From $M_Q\varphi \ll (M_{s,Q}\varphi^*)^* \leqslant (M_Q\varphi^*)^*$ we deduce $M_Q\varphi \ll (M_Q\varphi^*)^*$. Now, Lemma 6.3 proves that M_q is continuous on $L^\varphi(\mathbb{R}^d)$. □

Remark 6.5. Let p be a bounded exponent on $\mathbb{R}^d$ with $1 < p^- \leqslant p^+ < \infty$ and let $\varphi(x,t) = t^{p(x)}$ for $t \in \mathbb{R}^{\geqslant 0}$, $x \in \mathbb{R}^d$ (see Example 2.3). Then φ is a proper N-function and Theorem 6.4 is applicable.

7. Characterization of (strong-)domination

In this section we will characterize the property of domination and strong domination in a "pointwise" sense, i.e. for proper N-functions φ, ψ on $\mathcal{X}^d$ with $\psi \lesssim \varphi$ or $\psi \ll \varphi$ we will estimate $\psi(Q,t)$ in terms of $\varphi(Q,t)$. We will need this characterization in Section 8 in order to show that domination equals strong domination in the context of generalized Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^d)$. We begin with a general lemma.

Lemma 7.1. *Let X be an arbitrary set. Let Y be a subset of the power set of X such that $M_1 \subset M_2 \in Y$ implies $M_1 \in Y$. Let $\varphi, \psi : X \to \mathbb{R}^{\geqslant 0}$. If there exists $A_1 > 0$ and $A_2, A_3 \geqslant 0$ such that for all $M \in Y$*

$$\sum_{\omega \in M} \varphi(\omega) \leqslant A_1 \quad \Rightarrow \quad \sum_{\omega \in M} \psi(\omega) \leqslant A_2 \sum_{\omega \in M} \varphi(\omega) + A_3, \tag{7.1}$$

then there exists $b : X \to \mathbb{R}^{\geqslant 0}$ *such that for all* $\omega \in X$ *holds*

$$\varphi(\omega) \leqslant \frac{A_1}{4} \quad \Rightarrow \quad \psi(\omega) \leqslant \max\left\{\frac{4A_3}{A_1}, 2A_2\right\}\varphi(\omega) + b(\omega) \tag{7.2}$$

and

$$\sup_{M \in Y} \sum_{\omega \in M} b(\omega) \leqslant A_3. \tag{7.3}$$

If on the other hand there exist $b : X \to \mathbb{R}^{\geqslant 0}$*,* $A_1 > 0$*, and* $A_2, A_3 \geqslant 0$ *such that* (7.2) *and* (7.3) *hold, then for all* $M \in Y$

$$\sum_{\omega \in M} \varphi(\omega) \leqslant \frac{A_1}{4} \quad \Rightarrow \quad \sum_{\omega \in M} \psi(\omega) \leqslant \max\left\{\frac{4A_3}{A_1}, 2A_2\right\} \sum_{\omega \in M} \varphi(\omega) + A_3. \tag{7.4}$$

Proof. Assume that $A_1 > 0$ and $A_2, A_3 \geqslant 0$ are such that (7.2) and (7.3) are satisfied. For $\omega \in X$, $\gamma, \delta > 0$ define

$$G(\omega, \gamma, \delta) := \begin{cases} \psi(\omega) - \frac{\gamma}{2}\varphi(\omega) & \text{if } \varphi(\omega) < \min\{\delta, \gamma^{-1}\psi(\omega)\}, \\ 0 & \text{else.} \end{cases}$$

Then $G(\omega, \gamma, \delta) \geqslant 0$.

Claim 1. *For all* $\omega \in X$ *there holds*

$$\varphi(\omega) \leqslant \delta \quad \Rightarrow \quad \psi(\omega) \leqslant \gamma\varphi(\omega) + G(\omega, \gamma, \delta). \tag{7.5}$$

Proof of Claim 1. We prove the claim by contradiction. Assume there exists $\omega \in X$ with $\varphi(\omega) \leqslant \delta$ and $\psi(\omega) > \gamma\varphi(\omega) + G(\omega, \gamma, \delta)$. Especially there holds $\psi(\omega) - \frac{\gamma}{2}\varphi(\omega) > G(\omega, \gamma, \delta)$. From this and the definition of $G(\omega, \gamma, \delta)$ we deduce $\varphi(\omega) \geqslant \min \delta, \gamma^{-1}\psi(\omega)$. Since $\varphi(\omega) \leqslant \delta$, this implies $\varphi(\omega) \geqslant \gamma^{-1}\psi(\omega)$. Thus $\psi(\omega) \leqslant \gamma\varphi(\omega) \leqslant \gamma\varphi(\omega) + G(\omega, \gamma, \delta)$ which contradicts the assumptions. □

Claim 2. *Let* $\delta_0 := A_1/4$*,* $\gamma_0 := \max\{4A_3/A_1, 2A_2\}$ *then*

$$\sup_{M \in Y} \sum_{\omega \in M} G(\omega, \gamma_0, \delta_0) \leqslant A_3. \tag{7.6}$$

Proof of Claim 2. We prove the claim by contradiction, so assume that (7.6) fails. Then there exists $M_0 \in Y$ such that

$$\sum_{\omega \in M_0} G(\omega, \gamma_0, \delta_0) > A_3.$$

Therefore there exists $M_1 \subset M_0$ and $\omega_0 \in M_1$ such that

$$G(\omega, \gamma_0, \delta_0) > 0 \quad \text{for all } \omega \in M_1, \tag{7.7}$$

$$\sum_{\omega \in M_1 \setminus \omega_0} G(\omega, \gamma_0, \delta_0) \leqslant A_3, \tag{7.8}$$

$$\sum_{\omega \in M_1} G(\omega, \gamma_0, \delta_0) > A_3. \tag{7.9}$$

Since $M_1 \subset M_0 \in Y$ there holds $M_1 \in Y$. From (7.7) we deduce that M_1 is at most countable and there holds

$$G(\omega, \gamma_0, \delta_0) = \psi(\omega) - \frac{\gamma_0}{2}\varphi(\omega) \quad \text{for all } \omega \in M_1, \tag{7.10}$$

$$\varphi(\omega) < \min\{\delta_0, \gamma_0^{-1}\psi(\omega)\} \quad \text{for all } \omega \in M_1. \tag{7.11}$$

This implies

$$\begin{aligned}\sum_{\omega\in M_1} \varphi(\omega) &\leqslant \delta_0 + \sum_{\omega\in M_1\setminus\omega_0} \gamma_0^{-1}\psi(\omega) \\ &= \delta_0 + \sum_{\omega\in M_1\setminus\omega_0} \gamma_0^{-1}\left(G(\omega,\gamma_0,\delta_0) + \frac{\gamma_0}{2}\varphi(\omega)\right) \\ &\leqslant \delta_0 + \gamma_0^{-1}A_3 + \frac{1}{2}\sum_{\omega\in M_1\setminus\omega_0} \varphi(\omega). \end{aligned} \tag{7.12}$$

Note that (7.12) remains true if we replace M_1 by an arbitrary finite subset $M \subset M_1$. For all such sets the last term is finite and can be absorbed by the left-hand side. By exhausting M_1 by finite subsets we can pass back to M_1. We get

$$\sum_{\omega\in M_1} \varphi(\omega) \leqslant 2\delta_0 + 2\gamma_0^{-1}A_3 \leqslant A_1. \tag{7.13}$$

On the other hand (7.9), (7.10), and $\gamma_0 \geqslant 2A_2$ imply

$$\sum_{\omega\in M_1} \psi(\omega) = \sum_{\omega\in M_1}\left(G(\omega,\gamma_0,\delta_0) + \frac{\gamma_0}{2}\varphi(\omega)\right) > A_3 + A_2 \sum_{\omega\in M_1}\varphi(\omega). \tag{7.14}$$

Now (7.13) and (7.14) contradict (7.1). This proves the claim. □

Let $b(Q) := G(Q, \gamma_0, \delta_0)$ then Claims 1 and 2 prove (7.2) and (7.3).

If on the other hand there exist $b : X \to \mathbb{R}^{\geqslant 0}$, $A_1 > 0$, and $A_2, A_3 \geqslant 0$ such that (7.2) and (7.3) hold, then (7.4) is obvious. □

Definition 7.2. For $b : \mathcal{X}^d \to \mathbb{R}^{\geqslant 0}$ we define

$$\|b\|_{\mathcal{Y}^d,1} := \sup_{\mathcal{Q}\in\mathcal{Y}^d} \sum_{Q\in\mathcal{Q}} |Q| b(Q), \qquad \|b\|_{\mathcal{Y}^d,\infty} := \sup_{Q\in\mathcal{Q}} |Q| b(Q).$$

Theorem 7.3. *Let φ, ψ be proper N-functions on $\mathcal{X}^d$ with $\varphi \lesssim \psi$, i.e. there exists $A_1 > 0$ and $A_2 \geqslant 0$ such that for all $\mathcal{Q} \in \mathcal{Y}^d$ and all sequences $\{t_Q\}_{Q\in\mathcal{Q}}$, $t_Q \geqslant 0$, holds*

$$\sum_{Q\in\mathcal{Q}} |Q|\varphi(Q, t_Q) \leqslant A_1 \quad \Rightarrow \quad \sum_{Q\in\mathcal{Q}} |Q|\psi(Q, t_Q) \leqslant A_2, \tag{7.15}$$

then there exists $b : \mathcal{X}^d \to \mathbb{R}^{\geqslant 0}$ such that

$$\|b\|_{\mathcal{Y}^d,1} \equiv \sup_{\mathcal{Q}\in\mathcal{Y}^d} \sum_{Q\in\mathcal{Q}} |Q| b(Q) \leqslant A_2 \tag{7.16}$$

and for all $Q \in \mathcal{X}^d$ *and all* $t \geqslant 0$ *there holds*

$$|Q|\varphi(Q,t) \leqslant \frac{A_1}{4} \quad \Rightarrow \quad \psi(Q,t) \leqslant \frac{4A_2}{A_1}\varphi(Q,t) + b(Q). \tag{7.17}$$

If on the other hand there exist $b : \mathcal{X}^d \to \mathbb{R}^{\geqslant 0}$ *and* $A_1, A_2 > 0$ *such that* (7.16) *and* (7.17) *hold, then*

$$\sum_{Q \in \mathcal{Q}} |Q|\varphi(Q,t_Q) \leqslant \frac{A_1}{4} \quad \Rightarrow \quad \sum_{Q \in \mathcal{Q}} |Q|\psi(Q,t_Q) \leqslant 2A_2, \tag{7.18}$$

i.e. $\varphi \lesssim \psi$.

Proof. Let $X := \mathcal{X}^d$ and $Y := \mathcal{Y}^d$, then X, Y are admissible for Lemma 7.1. For $u : \mathcal{X}^d \to \mathbb{R}^{\geqslant 0}$ define $\varphi_u, \psi_u : \mathcal{X}^d \to \mathbb{R}^{\geqslant 0}$ by

$$\varphi_u(Q) := |Q|\varphi\big(Q, u(Q)\big), \quad \psi_u(Q) := |Q|\psi\big(Q, u(Q)\big).$$

Then (7.15) implies for all $\mathcal{Q} \in \mathcal{Y}^d$

$$\sum_{Q \in \mathcal{Q}} \varphi_u(Q) \leqslant A_1 \quad \Rightarrow \quad \sum_{Q \in \mathcal{Q}} \psi_u(Q) \leqslant A_2. \tag{7.19}$$

Thus we can apply Lemma 7.1 to $X := \mathcal{X}^d$, $Y := \mathcal{Y}^d$, and φ_u, ψ_u, i.e. there exists $a_u : \mathcal{X}^d \to \mathbb{R}^{\geqslant 0}$ such that for all $Q \in \mathcal{X}^d$ holds

$$\varphi_u(Q) \leqslant \frac{A_1}{4} \quad \Rightarrow \quad \psi_u(Q) \leqslant \frac{4A_2}{A_1}\varphi_u(Q) + a_u(Q)$$

and

$$\sup_{\mathcal{Q} \in \mathcal{Y}^d} \sum_{Q \in \mathcal{Q}} a_u(Q) \leqslant A_2.$$

Thus for all $Q \in \mathcal{X}^d$

$$|Q|\varphi\big(Q, u(Q)\big) \leqslant \frac{A_1}{4} \quad \Rightarrow \quad |Q|\psi\big(Q, u(Q)\big) \leqslant \frac{4A_2}{A_1}|Q|\varphi\big(Q, u(Q)\big) + a_u(Q)$$

and

$$\sup_{Q \in \mathcal{Q}} \sum_{Q \in \mathcal{Q}} a_u(Q) \leqslant A_2. \tag{7.20}$$

Define $b : \mathcal{X}^d \times \mathbb{R}^{\geqslant 0}$ by

$$b(Q,t) := \begin{cases} |Q|^{-1} \inf\limits_{\substack{u : \mathcal{X}^d \to \mathbb{R}^{\geqslant 0} \\ \text{with } u(Q) = t}} a_u(Q) & \text{if } |Q|\varphi(Q,t) \leqslant A_1/4, \\ 0 & \text{else.} \end{cases}$$

Then for all $Q \in \mathcal{X}^d$ and all $t \geqslant 0$

$$|Q|\varphi(Q,t) \leqslant \frac{A_1}{4} \quad \Rightarrow \quad \psi(Q,t) \leqslant \frac{4A_2}{A_1}\varphi(Q,t) + b(Q,t) \tag{7.21}$$

and for all $\mathcal{Q} \in \mathcal{Y}^d$ and all sequences $\{t_Q\}_{Q\in\mathcal{Q}}$, $t_Q \geqslant 0$, holds

$$\sup_{\mathcal{Q}\in\mathcal{Q}} \sum_{Q\in\mathcal{Q}} |Q| b(Q, t_Q) \leqslant A_2. \tag{7.22}$$

Define $b : \mathcal{X}^d \to \mathbb{R}^{\geqslant 0}$ by

$$b(Q) := \sup_{t\geqslant 0} b(Q, t).$$

Then (7.20), (7.21) and (7.22) imply

$$|Q|\varphi(Q, t) \leqslant \frac{A_1}{4} \quad \Rightarrow \quad \psi(Q, t) \leqslant \frac{4A_2}{A_1}\varphi(Q, t) + b(Q)$$

and

$$\sup_{\mathcal{Q}\in\mathcal{Q}} \sum_{Q\in\mathcal{Q}} |Q| b(Q) \leqslant A_2.$$

This proves (7.17) and (7.16).

If on the other hand there exists $b : \mathcal{X}^d \to \mathbb{R}^{\geqslant 0}$ and $A_1, A_2 > 0$ such that (7.16) and (7.17) hold then (7.18) is obvious. The strong Δ_2-condition for φ and ψ implies that $\varphi \lesssim \psi$.

Theorem 7.4. *Let $\varphi, \psi : \mathcal{X}^d \times \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}$ be such that $\psi \ll \varphi$, i.e. for every $A_1 > 0$ exists $A_2 > 0$ such that* (6.1) *and* (6.2) *imply* (6.3). *Then there exists $b : \mathcal{X}^d \times \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}$ such that for all $Q \in \mathcal{X}^d$ and $\lambda > 0$*

$$|Q|\varphi(Q, 4\lambda) \leqslant \frac{A_1}{4} \quad \Rightarrow \quad \psi(Q, \lambda) \leqslant \frac{4A_2}{A_1}\varphi(Q, 4\lambda) + b(Q, \lambda). \tag{7.23}$$

Moreover, for all $\mathcal{Q} \in \mathcal{Y}^d$ and all sequences $\{t_Q\}_{Q\in\mathcal{Q}}$ with $t_Q \geqslant 0$ there holds

$$\sup_{\mathcal{Q}\in\mathcal{Y}^d} \sum_{Q\in\mathcal{Q}} |Q| b(Q, t_Q) \leqslant A_2, \tag{7.24}$$

$$\int_0^\infty \lambda^{-1} \sup_{\mathcal{Q}\in\mathcal{Y}^d} \sum_{Q\in\mathcal{Q}} |Q| b(Q, \lambda)\, \mathrm{d}\lambda \leqslant A_2. \tag{7.25}$$

If on the other hand there exist $b : \mathcal{X}^d \to \mathbb{R}^{\geqslant 0}$ and $A_1, A_2 > 0$ such that (7.23), (7.24), *and* (7.25) *hold then*

$$\sum_{Q\in\mathcal{Q}_\lambda} |Q|\varphi(Q, \lambda) \leqslant \frac{A_1}{4} \quad \textit{and} \quad \int_0^\infty \lambda^{-1} \sum_{Q\in\mathcal{Q}_\lambda} |Q|\varphi(Q, \lambda)\, \mathrm{d}\lambda \leqslant \frac{A_1}{4}$$

implies

$$\sum_{Q\in\mathcal{Q}_\lambda} |Q|\psi(Q, \lambda) \leqslant 2A_2 \quad \textit{and} \quad \int_0^\infty \lambda^{-1} \sum_{Q\in\mathcal{Q}_\lambda} |Q|\psi(Q, \lambda)\, \mathrm{d}\lambda \leqslant 2A_2,$$

i.e. $\psi \ll \varphi$.

Proof. Let $X := \mathbb{Z} \times \mathcal{X}^d$. Define $\pi_k : X \to \mathcal{X}^d$ by

$$\pi_k(M) := \big\{Q \in \mathcal{X}^d \colon (k, Q) \in M\big\}.$$

Further let

$$Y := \big\{M \subset (\mathbb{Z} \times \mathcal{X}^d) \colon \pi_k(M) \in \mathcal{Y}^d \text{ for all } k \in \mathbb{Z}\big\}.$$

Then X, Y are admissible for Lemma 7.1.

Claim 1. *For all $M \in Y$ holds*

$$\sum_{(k,Q)\in M} |Q|\varphi(Q, 2^k) \leqslant A_1 \quad \Rightarrow \quad \sum_{(k,Q)\in M} |Q|\psi(Q, 2^{k-1}) \leqslant 2A_2.$$

Proof of Claim 1. Let $M \in Y$ be such that $\sum_{(k,Q)\in M} |Q|\varphi(Q, 2^k) \leqslant A_1$. Let $\{\mathcal{Q}_\lambda\}_{\lambda>0}$, $\mathcal{Q}_\lambda \in \mathcal{Y}^d$, be defined by

$$\mathcal{Q}_\lambda := \pi_k(M) \quad \text{for } 2^{k-1} < \lambda \leqslant 2^k.$$

Since $\varphi(Q, t)$ is non-decreasing in t, there holds for all $k \in \mathbb{Z}$ and $2^{k-1} < \lambda \leqslant 2^k$

$$\sum_{Q\in\mathcal{Q}_\lambda} |Q|\varphi(Q, \lambda) \leqslant \sum_{Q\in\pi_k(M)} |Q|\varphi(Q, 2^k) \leqslant A_1. \tag{7.26}$$

Moreover,

$$\begin{aligned}\int_0^\infty \lambda^{-1} \sum_{Q\in\mathcal{Q}_\lambda} |Q|\varphi(Q, \lambda)\,\mathrm{d}\lambda &\leqslant \sum_{k\in\mathbb{Z}} \int_{2^{k-1}}^{2^k} \lambda^{-1} \sum_{Q\in\pi_k(M)} |Q|\varphi(Q, 2^k)\,\mathrm{d}\lambda \\ &= (\ln 2) \sum_{(k,Q)\in M} |Q|\varphi(Q, 2^k) \leqslant A_1.\end{aligned} \tag{7.27}$$

Thus $\{\mathcal{Q}_\lambda\}_{\lambda>0}$ satisfies (6.1) and (6.2). Hence (6.3) implies

$$\int_0^\infty \lambda^{-1} \sum_{Q\in\mathcal{Q}_\lambda} |Q|\psi(Q, \lambda)\,\mathrm{d}\lambda \leqslant A_2.$$

Since $\psi(Q, t)$ is non-decreasing in t, this implies

$$\begin{aligned}\sum_{(k,Q)\in M} |Q|\psi(Q, 2^{k-1}) &\leqslant 2\sum_{k\in\mathbb{Z}} \int_{2^{k-1}}^{2^k} \lambda^{-1} \sum_{Q\in\pi_k(M)} |Q|\varphi(Q, 2^{k-1})\,\mathrm{d}\lambda \\ &\leqslant 2\int_0^\infty \lambda^{-1} \sum_{Q\in\mathcal{Q}_\lambda} |Q|\varphi(Q, \lambda)\,\mathrm{d}\lambda \leqslant 2A_2.\end{aligned}$$

This proves the claim.

From Lemma 7.1 and Claim 1 we deduce that there exist a: $\mathbb{Z}\times\mathcal{X}^d$ such that for all $Q\in\mathcal{X}^d$ and all $k\in\mathbb{Z}$

$$|Q|\varphi(Q,2^k)\leqslant\frac{A_1}{4}\quad\Rightarrow\quad|Q|\psi(Q,2^{k-1})\leqslant\frac{8A_2}{A_1}|Q|\varphi(Q,2^k)+a(k,Q)$$

and

$$\sup_{M\in Y}\sum_{(k,M)\in M}a(k,Q)\leqslant 2A_2.$$

From the definition of Y we deduce

$$\sum_{k\in\mathbb{Z}}\sup_{\mathcal{Q}\in\mathcal{Y}^d}\sum_{Q\in\mathcal{Q}}a(k,Q)\leqslant 2A_2. \tag{7.28}$$

Define b: $\mathcal{X}^d\times\mathbb{R}^{>0}$ by

$$b(Q,\lambda):=|Q|^{-1}a(k+1,Q)\qquad\text{for } 2^{k-1}<\lambda\leqslant 2^k.$$

We will now prove (7.23). Let $Q\in\mathcal{Q}$ and $\lambda>1$ such that

$$|Q|\varphi(Q,4\lambda)\leqslant\frac{A_1}{4} \tag{7.29}$$

and let k be such that $2^{k-1}<\lambda\leqslant 2^k$. Then $|Q|\varphi(Q,2^{k+1})\leqslant A_1/4$ and therefore

$$\begin{aligned}|Q|\psi(Q,\lambda)&\leqslant|Q|\psi(Q,2^k)\\&\leqslant\frac{8A_2}{A_1}|Q|\varphi(Q,2^{k+1})+a(k+1,Q)\\&=\frac{8A_2}{A_1}|Q|\varphi(Q,4\lambda)+b(Q,\lambda)\end{aligned} \tag{7.30}$$

where we have used again that $\varphi(Q,t)$, $\psi(Q,t)$ are non-decreasing in t. Now (7.29) and (7.30) prove (7.23). Furthermore for any family $\{\mathcal{Q}_\lambda\}_\lambda>0$, $\mathcal{Q}_\lambda\in\mathcal{Y}^d$ holds

$$\begin{aligned}\int_0^\infty\lambda^{-1}\sum_{Q\in\mathcal{Q}_\lambda}b(Q,\lambda)\,\mathrm{d}\lambda&=\sum_{k\in\mathbb{Z}}\int_{2^{k-1}}^{2^k}\lambda^{-1}\sum_{Q\in\mathcal{Q}_\lambda}a(k+1,Q)\,\mathrm{d}\lambda\\&\leqslant\sum_{k\in\mathbb{Z}}\sup_{\mathcal{Q}\in\mathcal{Y}^d}\sum_{Q\in\mathcal{Q}}a(k+1,Q)\\&\overset{(7.28)}{\leqslant}2A_2.\qquad\square\end{aligned}$$

The following remark will specify the difference of $\psi\lesssim\varphi$ and $\psi\ll\varphi$.

Remark 7.5. Let φ,ψ proper N-function on $\mathcal{X}^d$ with $\psi\lesssim\varphi$. Let A_1,A_2,b be as in Theorem 7.3. Define $\tilde{b}:\mathcal{X}^d\times\mathbb{R}^{\geqslant0}\to\mathbb{R}^{\geqslant0}$ by

$$\tilde{b}(Q,t):=\begin{cases}\psi(Q,t) & \text{if } \psi(Q,t)\leqslant b(Q),\\ 0 & \text{else.}\end{cases} \tag{7.31}$$

Then $\tilde{b}$ satisfies (7.23) and (7.24) (with b replaced by $\tilde{b}$). On the other hand (2.3) implies

$$\int_0^\infty \lambda^{-1}\tilde{b}(Q,\lambda)\,\mathrm{d}\lambda \leqslant \int_0^{\psi^{-1}(Q,b(Q))} \psi'(Q,\lambda)\,\mathrm{d}\lambda = b(Q).$$

Thus instead of (7.25) we only obtain

$$\sup_{\mathcal{Q}\in\mathcal{Y}^d}\int_0^\infty \lambda^{-1}\sum_{Q\in\mathcal{Q}}|Q|\tilde{b}(Q,\lambda)\,\mathrm{d}\lambda \leqslant \sup_{\mathcal{Q}\in\mathcal{Y}^d}\sum_{Q\in\mathcal{Q}}|Q|b(Q)\leqslant A_2,$$

i.e. in comparison to (7.25) the supremum is taken outside the integral. This is the precise difference of domination and strong domination.

Remark 7.6. Let us remark that in the case of classical weighted Lebesgue spaces, i.e. $\varphi(x,t)=t^p\omega(x)$, domination immediately implies strong domination. Indeed, let φ be of class $\mathcal{A}$ and let $s\geqslant 1$ such that $M_Q\varphi \lesssim (M_{s,Q}\varphi^*)^*$. For $Q\in\mathcal{X}^d$ let $t_{0,Q}:=1/\|\chi_Q\|_\varphi$ then from Lemma 8.3 (see below) it follows that uniformly in $t>0$ and $Q\in\mathcal{X}^d$

$$\begin{aligned}(M_Q\varphi)(t) &\sim \left(\frac{t}{t_{0,Q}}\right)^p (M_Q\varphi)(t_{0,Q}) \sim \left(\frac{t}{t_{0,Q}}\right)^p \\ &\sim \left(\frac{t}{t_{0,Q}}\right)^p (M_{s,Q}\varphi^*)^*(t_{0,Q}) \sim (M_{s,Q}\varphi^*)^*(t).\end{aligned}$$

Thus (7.17), (7.23) hold for some $A_2>0$ with $b:=0$. Now Remark 7.5 implies the equivalence of $M_Q\varphi\lesssim (M_{s,Q}\varphi^*)^*$ and $M_Q\varphi \ll (M_{s,Q}\varphi^*)^*$.

8. Generalized Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^d)$

In Lemma 3.2 we have seen that it is necessary for the continuity of M on $L^\varphi(\mathbb{R}^d)$ that φ is of class $\mathcal{A}$. From class $\mathcal{A}$ we have deduced in Theorem 5.7 that $M_{s_2,Q}\varphi\lesssim (M_{s_2,Q}\varphi^*)^*$ for some $s_2>1$, especially $M_Q\varphi\lesssim (M_{s_2,Q}\varphi^*)^*$. On the other hand we know from Theorem 6.4 and Remark 6.5 that if $M_Q\varphi \ll (M_{s_1,Q}\varphi^*)^*$ for some $s_1>1$, then M and even M_q for some $q>1$ is continuous on $L^\varphi(\mathbb{R}^d)$. Due to this little gap we do not know yet if the necessary condition "φ is of class $\mathcal{A}$" is also sufficient for the continuity of M or M_q. In this section we will show that in the case of generalized Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^d)$ the condition "φ is of class $\mathcal{A}$" is sufficient for the continuity of M and even M_q for some $q>1$. Indeed, we will show that $M_{s_2,Q}\varphi\lesssim (M_{s_2,Q}\varphi^*)^*$ for some $s_2>1$ implies $M_{s_1,Q}\varphi \ll (M_{s_1,Q}\varphi^*)^*$ for some $1<s_1<s_2$, especially $M_Q\varphi\ll (M_{s_1,Q}\varphi^*)^*$. This shows that "class $\mathcal{A}$" is necessary and sufficient in the case of $L^{p(\cdot)}(\mathbb{R}^d)$. The case of general proper N-functions φ remains open. The main result of this section is the following.

Theorem 8.1. *Let p be a bounded exponent on $\mathbb{R}^d$ with $1<p^-\leqslant p^+<\infty$ and let $\varphi(x,t)=t^{p(x)}$ for all $t\in\mathbb{R}^{\geqslant 0}$ and all $x\in\mathbb{R}^d$* (*see Example* 2.3)*. The following are equivalent*

(i) *φ is of class $\mathcal{A}$.*
(ii) *M is continuous on $L^{p(\cdot)}(\mathbb{R}^d)$.*
(iii) *M_q is continuous on $L^{p(\cdot)}(\mathbb{R}^d)$ for some $q > 1$ ("left-openness").*
(iv) *M is continuous on $L^{\frac{p(\cdot)}{q}}(\mathbb{R}^d)$ for some $q > 1$ ("left-openness").*
(v) *$\psi(x,t) := t^{p'(x)}$ is of class $\mathcal{A}$.*
(vi) *M is continuous on $L^{p'(\cdot)}(\mathbb{R}^d)$.*

Before we get to the proof of Theorem 8.1 we will need some auxiliary results. Also note that we will provide some fundamental applications of Theorem 8.1 at the end of this section.

Lemma 8.2. *Let $\rho_j : \mathcal{X}^d \times \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}$, $j = 1, 2$ be proper N-function with $\rho_1 \lesssim \rho_2 \lesssim \rho_1$. Then for all $d_1, D_1 > 0$ there exist $d_2, D_2 > 0$ such that the following holds: If $\mathcal{Q} \in \mathcal{Y}^d$ and $\{t_Q\}_{Q\in\mathcal{Q}}$ with $t_Q \geqslant 0$ satisfy*

$$d_1 \leqslant \sum_{Q\in\mathcal{Q}} |Q|\rho_1(t_Q) \leqslant D_1 \tag{8.1}$$

then

$$d_2 \leqslant \sum_{Q\in\mathcal{Q}} |Q|\rho_2(t_Q) \leqslant D_2. \tag{8.2}$$

Proof. Since ρ_1 and ρ_2 satisfy the strong Δ_2-condition it suffices to prove the case $d_1 = D_1 = 1$. Let $A_2 > 0$ be such that (3.1) holds for $\rho_2 \lesssim \rho_1$ and $\rho_1 \lesssim \rho_2$ with $A_1 := 1$. Further let $C_0 > 0$ be such that $\rho_2(Q, 2t) \leqslant C_0\rho_2(Q,t)$ for all $Q \in \mathcal{X}^d$ and $t \geqslant 0$. Let $\mathcal{Q} \in \mathcal{Y}^d$ and $\{t_Q\}_{Q\in\mathcal{Q}}$ with $t_Q \geqslant 0$ be such that (8.1) holds. Then the second inequality of (8.2) holds with $D_2 := A_2$. Let $m \in N$ such that $2^m \geqslant A_2$ and let $d_2 := C_0^{-m}$. We proceed by contradiction. Assume that $\sum_{Q\in\mathcal{Q}} |Q|\rho_2(t_Q) < d_2$. Then by convexity

$$\sum_{Q\in\mathcal{Q}} |Q|\rho_2(2^m t_Q) \leqslant C_0^m \sum_{Q\in\mathcal{Q}} |Q|\rho_2(t_Q) < C_0^m d_2 = 1,$$
$$\sum_{Q\in\mathcal{Q}} |Q|\rho_1(2^m t_Q) \geqslant 2^m \sum_{Q\in\mathcal{Q}} |Q|\rho_1(t_Q) = 2^m \geqslant A_2.$$

This contradicts the choice of A_1, A_2 for $\rho_1 \lesssim \rho_2$. This proves the lemma. $\square$

Lemma 8.3. *Let φ be a proper N-function on $\mathbb{R}^d$ with $(M_{s,Q}\varphi) \lesssim (M_{s,Q}\varphi^*)^*$ for some $s \geqslant 1$. Then uniformly in $Q \in \mathcal{X}^d$*

$$|Q|(M_{s,Q}\varphi)\left(\frac{1}{\|\chi_Q\|_\varphi}\right) \sim 1, \qquad |Q|(M_{s,Q}\varphi^*)^*\left(\frac{1}{\|\chi_Q\|_\varphi}\right) \sim 1. \tag{8.3}$$

Proof. For $Q \in \mathcal{X}^d$ define $t_{0,Q} := 1/\|\chi_Q\|_\varphi$. Then

$$|Q|(M_Q\varphi)(t_{0,Q}) = \int_Q \varphi\left(\frac{1}{\|\chi_Q\|_\varphi}\right) \mathrm{d}x = 1. \tag{8.4}$$

Since $M_Q\varphi \leqslant M_{s,Q}\varphi$ and $(M_{s,Q}\varphi^*)^* \leqslant (M_Q\varphi^*)^*$, by Jensen's inequality, and $(M_Q\varphi^*)^* \lesssim M_Q\varphi$ by (3.3), there holds

$$M_Q\varphi \lesssim \rho \lesssim M_Q\varphi, \tag{8.5}$$

where $\rho : \mathcal{X}^d \times \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}$ is either $M_{s,Q}\varphi$ or $(M_{s,Q}\varphi^*)^*$. Thus (8.4) and Lemma 8.2 prove the lemma. □

Lemma 8.4. *Let p, φ be as in Theorem* 8.1. *Further assume $M_{s,Q}\varphi \lesssim (M_{s,Q}\varphi^*)^*$ for some $s \geqslant 1$. Define $\alpha_s : \mathcal{X}^d \times \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ by*

$$\alpha_s(Q,t) := \frac{(M_{s,Q}\varphi)(t)}{(M_{s,Q}\varphi^*)^*(t)}. \tag{8.6}$$

Then uniformly in $Q \in \mathcal{X}^d$ and $t > 0$

$$\alpha_s\left(Q, \frac{1}{\|\chi_Q\|\varphi}\right) \sim 1, \qquad \alpha_s(Q,1) \sim 1. \tag{8.7}$$

Moreover, there exists $C_5 \geqslant 1$ such that for all $Q \in \mathcal{X}^d$

$$\alpha_s(Q,t_2) \leqslant C_5\big(\alpha_s(Q,t_1)+1\big) \quad \text{for } 0 < t_1 \leqslant t_2 \leqslant 1, \tag{8.8}$$

$$\alpha_s(Q,t_3) \leqslant C_5\big(\alpha_s(Q,t_4)+1\big) \quad \text{for } 1 \leqslant t_3 \leqslant t_4. \tag{8.9}$$

Furthermore, for all $C_6, C_7 > 0$ there exists $C_8 \geqslant 1$ such that for all $Q \in \mathcal{X}^d$

$$t \in \left[C_6 \min\left\{1, \frac{1}{\|\chi_Q\|_\varphi}\right\}, C_7 \max\left\{1, \frac{1}{\|\chi_Q\|_\varphi}\right\}\right] \quad \Rightarrow \quad \alpha_s(Q,t) \leqslant C_8. \tag{8.10}$$

Proof. The first part of (8.7) follows from Lemma 8.3. Define $a : \mathcal{X}^d \times \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}$ by $a := \varphi'$, then $(\varphi^*)' = a^{-1}$. Due to (2.4) (applied to $(M_{s,Q}\varphi^*)$) and the strong Δ_2 condition there holds

$$\begin{aligned}(M_{s,Q}\varphi^*)^*\big((M_{s,Q}a^{-1})(t)\big) \ &\sim \ (M_{s,Q}\varphi^*)^*\left(\frac{M_{s,Q}\varphi^*(t)}{t}\right)\\ &\overset{(2.4)}{\sim} \ (M_{s,Q}\varphi^*)(t) \sim t(M_{s,Q}a^{-1})(t).\end{aligned} \tag{8.11}$$

Thus

$$\begin{aligned}\alpha_s\big(Q,(M_{s,Q}a^{-1})(t)\big) \ &= \ \frac{(M_{s,Q}\varphi)((M_{s,Q}a^{-1})(t))}{(M_{s,Q}\varphi^*)^*((M_{s,Q}a^{-1})(t))}\\ &\overset{(8.11)}{\sim} \ \frac{(M_{s,Q}\varphi)((M_{s,Q}a^{-1})(t))}{t(M_{s,Q}a^{-1})(t)} \sim \frac{(M_{s,Q}a)((M_{s,Q}a^{-1})(t))}{t}.\end{aligned}$$

Moreover, $a(x,t) \sim t^{p(x)-1}$ and $a^{-1}(x,t) \sim t^{\frac{1}{p(x)-1}}$ so

$$\alpha_s\big(Q,(M_{s,Q}a^{-1})(t)\big) \sim \left(\fint_Q \left(\fint_Q t^{\frac{s(p(y)-p(z))}{(p(z)-1)(p(y)-1)}}\,\mathrm{d}z\right)^{1/(p(y)-1)} \mathrm{d}y\right)^{1/s}. \tag{8.12}$$

Since $(M_{s,Q}a^{-1})(1) \sim 1$ this proves (8.7). Define

$$\beta_s^{>}(Q,t) := \biggl(\fint_Q \biggl(\fint_Q t^{\frac{s(p(y)-p(z))}{(p(z)-1)(p(y)-1)}}\,\mathrm{d}z\biggr)^{1/(p(y)-1)} \chi_{p(y)>p(z)}\,\mathrm{d}y\biggr)^{1/s},$$

$$\beta_s^{\leqslant}(Q,t) := \biggl(\fint_Q \biggl(\fint_Q t^{\frac{s(p(y)-p(z))}{(p(z)-1)(p(y)-1)}}\,\mathrm{d}z\biggr)^{1/(p(y)-1)} \chi_{p(y)\leqslant p(z)}\,\mathrm{d}y\biggr)^{1/s},$$

then

$$\begin{aligned}
&\alpha_s\bigl(Q,(M_{s,Q}a^{-1})(t)\bigr) \sim \beta_s^{>}(Q,t) + \beta_s^{\leqslant}(Q,t) \quad \text{uniformly in } Q,t\\
&0 \leqslant \beta_s^{>}(Q,t) \leqslant 1 \quad \text{for } 0 < t \leqslant 1,\\
&0 \leqslant \beta_s^{\leqslant}(Q,t) \leqslant 1 \quad \text{for } t \geqslant 1,\\
&\beta_s^{>}(Q,t) \text{ is monotonously increasing on } [1,\infty),\\
&\beta_s^{\leqslant}(Q,t) \text{ is monotonously decreasing on } (0,1].
\end{aligned}$$

Thus there exists $C_5 \geqslant 1$ such that

$$\begin{aligned}
&\alpha_s\bigl(Q,(M_{s,Q}a^{-1})(t_2)\bigr) \leqslant C_5\bigl(\alpha_s\bigl(Q,(M_{s,Q}a^{-1})(t_1)\bigr)+1\bigr) \quad \text{for } 0 < t_1 \leqslant t_2 \leqslant 1,\\
&\alpha_s\bigl(Q,(M_{s,Q}a^{-1})(t_3)\bigr) \leqslant C_5\bigl(\alpha_s\bigl(Q,(M_{s,Q}a^{-1})(t_4)\bigr)+1\bigr) \quad \text{for } 1 \leqslant t_3 \leqslant t_4.
\end{aligned}$$

This, $(M_{s,Q}a^{-1})(1) \sim 1$, and the strong Δ_2-condition prove (8.8) and (8.9). Now (8.10) follows immediately from (8.7), (8.8), (8.9), and the Δ_2-condition. This proves the lemma. □

Lemma 8.5. *Let p, φ, s be as in Lemma* 8.4*, especially $(M_{s,Q}\varphi) \lesssim (M_{s,Q}\varphi^*)^*$. Then there exists $b : \mathcal{X}^d \to \mathbb{R}^{\geqslant 0}$ and $K > 0$ such that $\|b\|_{\mathcal{Y}^d,1} + \|b\|_{\mathcal{Y}^d,\infty} < \infty$ (see Definition* 7.2*) and for all $Q \in \mathcal{X}^d$ and all $t \geqslant 0$ holds*

$$|Q|(M_{s,Q}\varphi^*)^*(t) \leqslant 1 \quad \Rightarrow \quad (M_{s,Q}\varphi)(t) \leqslant K(M_{s,Q}\varphi^*)^*(Q,t) + b(Q). \tag{8.13}$$

Moreover, for all $Q \in \mathcal{X}^d$ and all $t \geqslant 1$ there holds

$$|Q|(M_{s,Q}\varphi^*)^*(t) \leqslant 1 \quad \Rightarrow \quad (M_{s,Q}\varphi)(t) \leqslant K(M_{s,Q}\varphi^*)^*(Q,t). \tag{8.14}$$

Proof. Due to Theorem 7.3 there exists $b_2 : \mathcal{X}^d \to \mathbb{R}^{\geqslant 0}$ with $\|b_2\|_{\mathcal{Y}^d,1} < \infty$ and $K_2 > 0$ such that for all $Q \in \mathcal{X}^d$ and all $t \geqslant 0$ holds

$$|Q|(M_{s,Q}\varphi^*)^*(t) \leqslant 1 \quad \Rightarrow \quad (M_{s,Q}\varphi)(t) \leqslant K_2(M_{s,Q}\varphi^*)^*(Q,t) + b_2(Q). \tag{8.15}$$

Assume that $|Q|(M_{s,Q}\varphi^*)^*(t) \leqslant 1$. Then due to Lemma 8.3 and the strong Δ_2-condition of $M_{s,Q}\varphi$ there exists $A \geqslant 0$ (independent of Q and t) such that $t \leqslant A/\|\chi_Q\|_\varphi$. Now due to by Lemma 8.4 there exists $C_8 \geqslant 1$ such that (8.10) holds for the choice $C_6 := 1$, $C_7 := A$. Let $K := \max\{C_8, K_2\}$ and define $b : \mathcal{X}^d \to \mathbb{R}^{\geqslant 0}$ by

$$b(Q) := \min\bigl\{(M_{s,Q}\varphi)(1), b_2(Q)\bigr\}. \tag{8.16}$$

Since $(M_{s,Q}\varphi)(1) \sim 1$, there holds $\|b\|_{\mathcal{Y}^d,1} + \|b\|_{\mathcal{Y}^d,\infty} < \infty$. If $0 \leqslant t \leqslant 1$ then by (8.15) and (8.16)

$$|Q|(M_{s,Q}\varphi^*)^*(t) \leqslant 1 \quad \Rightarrow \quad (M_{s,Q}\varphi)(t) \leqslant K_2(M_{s,Q}\varphi^*)^*(Q,t) + b(Q). \tag{8.17}$$

If on the other hand $1 < t \leqslant A/\|\chi_Q\|_\varphi$, then by (8.10) we deduce $\alpha_s(Q,t) \leqslant C_8$. The definition of α_s and $C_8 \leqslant K$ immediately imply (8.14). This proves the lemma. □

Lemma 8.6. *Let p, φ be as in Theorem* 8.1. *Further assume $M_{s_2,Q}\varphi \lesssim (M_{s_2,Q}\varphi^*)^*$ for some $1 \leqslant s_1 \leqslant s_2$. Let $\alpha_{s_1}, \alpha_{s_2}$ be defined as in* (8.6). *Then uniformly in $Q \in \mathcal{X}^d$ and $t > 0$*

$$\big(\alpha_{s_2}(Q, t^{s_1/s_2})\big)^{s_2/s_1} \sim \alpha_{s_1}(Q,t).$$

Proof. From $\varphi(x,t) = t^{p(x)}$ we deduce $\varphi^*(x,t) = \beta(x)t^{p'(x)}$ where $\beta(x) := (p(x) - 1)p(x)^{-p'(x)}$. Note that for any $r \geqslant 1$ there holds $\varphi(x,t^r) \sim \varphi(x,t)^r$ and $\varphi^*(x,t^r) \sim (\varphi^*(x,t))^r$ uniformly in $x \in \mathbb{R}^d$, $t \geqslant 0$. Thus

$$(M_{s_1,Q}\varphi)(t) \sim \big((M_Q\varphi)(t^{s_1})\big)^{1/s_1} \sim \big((M_{s_2,Q}\varphi)(t^{s_1/s_2})\big)^{s_2/s_1}, \tag{8.18}$$

$$(M_{s_1,Q}\varphi^*)(t) \sim \big((M_Q\varphi^*)(t^{s_1})\big)^{1/s_1} \sim \big((M_{s_2,Q}\varphi^*)(t^{s_1/s_2})\big)^{s_2/s_1}. \tag{8.19}$$

uniformly in $Q \in \mathcal{X}^d$, $t \geqslant 0$. From $\varphi^*(x,t) = \beta(x)t^{p'(x)}$ we easily deduce that $(Q,t) \mapsto ((M_{s_2,Q}\varphi^*)(t^{s_1/s_2}))^{s_2/s_1}$ is a proper N-function. Additionally, we deduce as in Lemma 3.4 that it is proper. Thus it follows from (8.19) and Lemma 5.8 that

$$(M_{s_1,Q}\varphi^*)^*(t) \sim \big((M_{s_2,Q}\varphi^*)^*(t^{s_1/s_2})\big)^{s_2/s_1}.$$

This and (8.18) imply

$$\big(\alpha_{s_2}(Q, t^{s_1/s_2})\big)^{s_2/s_1} \sim \alpha_{s_1}(Q,t).$$

This proves the lemma. □

We will come to the key lemma from which we will derive Theorem 8.1.

Lemma 8.7. *Let p, φ be as in Theorem* 8.1. *Further assume that φ is of class $\mathcal{A}$. Then there exists $s_1 > 1$ such that $(M_{s_1,Q}\varphi) \ll (M_{s_1,Q}\varphi^*)^*$.*

Proof. Due to Theorem 5.7 there exists $s_2 > 1$ with

$$M_{s_2,Q}\varphi \lesssim M_Q\varphi \lesssim (M_Q\varphi^*)^* \lesssim (M_{s_2,Q}\varphi^*)^*. \tag{8.20}$$

Let s_1 be such that $1 < s_1 < s_2$ then by (8.20) and Jensen's inequality holds $M_{s_1,Q}\varphi \lesssim (M_{s_1,Q}\varphi^*)^*$. Due to Lemma 8.5 there exists $b_2 : \mathcal{X}^d \to \mathbb{R}^{\geqslant 0}$ with $\|b_2\|_{\mathcal{Y}^d,1} + \|b_2\|_{\mathcal{Y}^d,\infty} < \infty$ and $K_2 \geqslant 1$ such that (8.13) and (8.14) hold (for the choice $s = s_2$ and $b = b_2$). Due to Lemma 8.3 and the strong Δ_2-condition of $(M_{s_2,Q}\varphi^*)^*$ there exists $0 < D_2 \leqslant 1$ (independent of Q and t) such that $t \leqslant D_2/\|\chi_Q\|_\varphi$ implies $|Q|(M_{s_2,Q}\varphi^*)^*(t) \leqslant 1$. Due to Lemma 8.4 there exists $C_8 \geqslant 1$ such that

$$t \in \left[D_2 \min\left\{1, \frac{1}{\|\chi_Q\|_\varphi}\right\}, \max\left\{1, \frac{1}{\|\chi_Q\|_\varphi}\right\}\right] \quad \Rightarrow \quad \alpha_{s_2}(Q,t) \leqslant C_8. \tag{8.21}$$

Moreover, by Lemma 8.6 and the strong Δ_2-condition there exists $A_0 \geqslant 1$ such that for all $Q \in \mathcal{X}^d$ and all $t > 0$

$$\alpha_{s_1}(Q,t) \leqslant A_0\big(\alpha_{s_2}(Q, t^{s_1/s_2})\big)^{s_2/s_1}. \tag{8.22}$$

Define $K_1 := A_0(\max\{2K_2, C_8\})^{s_2/s_1}$.

Claim 2. *For all $Q \in \mathcal{X}^d$ and $t > 0$ with*

$$|Q|(M_{s_1,Q}\varphi^*)^*(t) \leqslant 1 \tag{8.23}$$

holds

$$(M_{s_1,Q}\varphi)(t) \leqslant \begin{cases} \max\{K_1(M_{s_1,Q}\varphi^*)^*(Q,t), 2b_2(Q)t^{1-s_1/s_2}\} & \text{for } 0 < t < 1, \\ K_1(M_{s_1,Q}\varphi^*)^*(Q,t) & \text{for } t \geqslant 1. \end{cases} \tag{8.24}$$

Proof of Claim 1. Assume that (8.23) is satisfied, then by Jensen's inequality

$$|Q|(M_{s_2,Q}\varphi^*)^*(t) \leqslant 1. \tag{8.25}$$

If $t \geqslant 1$, then by (8.14) and Jensen's inequality

$$(M_{s_1,Q}\varphi)(t) \leqslant (M_{s_2,Q}\varphi)(t) \leqslant K_2(M_{s_2,Q}\varphi)(t) \leqslant K_1(M_{s_1,Q}\varphi^*)^*(Q,t),$$

so (8.24) holds in this case. If $0 < t < 1$ and $\alpha_{s_1}(Q,t) \leqslant K_1$, then

$$|Q|(M_{s_1,Q}\varphi)(t) = \alpha_{s_1}(Q,t)|Q|(M_{s_1,Q}\varphi^*)^*(t) \leqslant K_1|Q|(M_{s_1,Q}\varphi^*)^*(t),$$

so (8.24) holds also in this case. It remains to consider the case

$$0 < t < 1 \quad \text{and} \quad \alpha_{s_1}(Q,t) > K_1.$$

From (8.22) we deduce

$$A_0\big(\max\{2K_2, C_8\}\big)^{s_2/s_1} = K_1 < \alpha_{s_1}(Q,t) \leqslant A_0\big(\alpha_{s_2}(Q, t^{s_1/s_2})\big)^{s_2/s_1}.$$

Especially

$$0 < t^{s_1/s_2} < 1 \quad \text{and} \quad \alpha_{s_2}(Q, t^{s_1/s_2}) > \max\{2K_2, C_8\}. \tag{8.26}$$

From (8.21) we deduce

$$0 < t^{s_1/s_2} < \frac{D_2}{\|\chi_Q\|_\varphi}.$$

Now the choice of D_2 implies

$$|Q|(M_{s_2,Q}\varphi^*)^*(t^{s_1/s_2}) \leqslant 1.$$

From (8.13) we deduce

$$(M_{s_2,Q}\varphi)(t^{s_1/s_2}) \leqslant K_2(M_{s_2,Q}\varphi^*)^*(t^{s_1/s_2}) + b_2(Q). \tag{8.27}$$

Since by (8.26) holds $\alpha_{s_2}(Q, t^{s_1/s_2}) \geqslant 2K_2$, we can absorb the first term of the right-hand side on the left-hand side, i.e.

$$(M_{s_2,Q}\varphi)(t^{s_1/s_2}) \leqslant 2b_2(Q). \tag{8.28}$$

It follows from $0 < t < 1$, $s_2 > s_1$, and Jensen's inequality that

$$\begin{aligned}(M_{s_1,Q}\varphi)(t) &\leqslant (M_{s_2,Q}\varphi)(t)\\ &\leqslant (M_{s_2,Q}\varphi)(t^{s_1/s_2})t^{1-s_1/s_2} \quad \text{by convexity of } \varphi\\ &\leqslant 2b_2(Q)t^{1-s_1/s_2} \quad \text{by (8.28),}\end{aligned}$$

so (8.24) holds also in this case. This proves the claim. □

We will now deduce from Claim 1 that $(M_{s_1,Q}\varphi) \ll (M_{s_1,Q}\varphi^*)^*$. Fix $A_1 := 1$. Let $\mathcal{Q}_\lambda \in \mathcal{Y}^d$, $\lambda > 0$, be such that (6.1) and (6.2) hold for $(M_{s_1,Q}\varphi^*)^*$. Then every pair $\lambda > 0$ and $Q \in \mathcal{Q}_\lambda$ satisfies (8.23). Thus by (8.24)

$$\begin{aligned}&\int_0^\infty \lambda^{-1} \sum_{Q\in\mathcal{Q}_\lambda} |Q|(M_{s_1,Q}\varphi)(\lambda)\,\mathrm{d}\lambda\\ &\quad\leqslant K_1\int_0^\infty \lambda^{-1} \sum_{Q\in\mathcal{Q}_\lambda} |Q|(M_{s_1,Q}\varphi^*)^*(\lambda)\,\mathrm{d}\lambda + 2\int_0^1 \sum_{Q\in\mathcal{Q}_\lambda} |Q|b_2(Q)\lambda^{-s_1/s_2}\,\mathrm{d}\lambda\\ &\quad\leqslant K_1 + \frac{s_2}{s_2-s_1}\|b_2\|_{\mathcal{Y}^d,1} =: A_2.\end{aligned}$$

This proves (6.3), i.e. $M_{s_1,Q}\varphi \ll (M_{s_1,Q}\varphi^*)^*$. This proves the lemma. □

We will now prove Theorem 8.1.

Proof of Theorem 8.1. (iii) ⇒ (ii): This follows from $Mf \leqslant M_q f$.

(ii) ⇒ (i): Follows from Lemma 3.2.

(i) ⇒ (iii): Let φ be of class $\mathcal{A}$, then by Lemma 8.7 exists $s_1 > 1$ such that $(M_{s_1,Q}\varphi) \ll (M_{s_1,Q}\varphi^*)^*$, especially $(M_Q\varphi) \ll (M_{s_1,Q}\varphi^*)^*$. Thus by Theorem 6.4 and Remark 6.5 there exists $q > 1$ such that M_q is continuous on $L^\varphi(\mathbb{R}^d) = L^{p(\cdot)}(\mathbb{R}^d)$.

(iii) ⇔ (iv): This follows from the identity

$$\|M_q f\|_{p(\cdot)} = \left\|\big(M\big(|f|^q\big)\big)^{1/q}\right\|_{p(\cdot)} = \left\|M\big(|f|^q\big)\right\|_{p(\cdot)/q}^{1/q}.$$

(i) ⇔ (v): Due to Example 2.3 we know that $L^\psi(\mathbb{R}^d) \equiv L^{p'(\cdot)}(\mathbb{R}^d) \cong L^{\varphi^*}(\mathbb{R}^d)$. Thus ψ is of class $\mathcal{A}$ if and only if φ^* is of class $\mathcal{A}$. The rest follows from Lemma 3.3.

(v) ⇔ (vi): This follows from the equivalence (i) ⇔ (ii) applied to ψ. □

8.1. Applications

In [8–10] M. Růžička and L. Diening have presented results on singular integrals and questions regarding fluid mechanics for the generalized Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^d)$. All these results are based on the sole requirement that M is continuous on the spaces $L^{p(\cdot)}(\mathbb{R}^d)$, $L^{p'(\cdot)}(\mathbb{R}^d)$, $L^{p(\cdot)/s}(\mathbb{R}^d)$, and $L^{(p(\cdot)/r)'}(\mathbb{R}^d)$ for some $0 < r < 1 < s$. Due to Theorem 8.1 these properties follow from class $\mathcal{A}$:

Corollary 8.8. *Let p be a bounded exponent on $\mathbb{R}^d$ with $1 < p^- \leqslant p^+ < \infty$ and let $\varphi(x,t) = t^{p(x)}$ for all $t \in \mathbb{R}^{\geqslant 0}$ and all $x \in \mathbb{R}^d$ (see Example 2.3). If φ is of class $\mathcal{A}$, then there exists $0 < r < 1 < s$ such that M is continuous on $L^{p(\cdot)}(\mathbb{R}^d)$, $L^{p'(\cdot)}(\mathbb{R}^d)$, $L^{p(\cdot)/s}(\mathbb{R}^d)$, and $L^{(p(\cdot)/r)'}(\mathbb{R}^d)$.*

Proof. The existence of $s > 1$ and the continuity of M on $L^{p(\cdot)}(\mathbb{R}^d)$, $L^{p'(\cdot)}(\mathbb{R}^d)$, and $L^{p(\cdot)/s}(\mathbb{R}^d)$ follow immediately from Theorem 8.1. Let $0 < r < 1$ be arbitrary. Then by continuity of M on $L^{p(\cdot)}(\mathbb{R}^d)$ and Jensen's inequality

$$\|Mf\|_{p(\cdot)/r} = \big\|(Mf)^{1/r}\big\|_{p(\cdot)}^r \leqslant \big\|M\big(|f|^{1/r}\big)\big\|_{p(\cdot)}^r \leqslant C\big\||f|^{1/r}\big\|_{p(\cdot)}^r = C\|f\|_{p(\cdot)/r}.$$

Thus M is continuous on $L^{p(\cdot)/r}(\mathbb{R}^d)$. Thus by Theorem 8.1 M is continuous on $L^{(p(\cdot)/r)'}(\mathbb{R}^d)$. This proves the corollary. □

As an immediate consequence of Corollary 8.8 we can state the results of [8] under weaker assumptions.

Definition 8.9. For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ define $f^\sharp : \mathbb{R}^d \to \mathbb{R}^{\geqslant 0} \cup \infty$ by

$$f^\sharp(x) := \sup_{Q \ni x} \fint_Q \big|f - (f)_Q\big| \,\mathrm{d}x,$$

where $(f)_Q := \fint_Q f \,\mathrm{d}y$ and the supremum is taken over all cubes $Q \in \mathcal{X}^d$ containing x.

Theorem 8.10. *Let p, φ be as is Corollary 8.8. If φ is of class $\mathcal{A}$, then there exists $A \geqslant 1$ such that for all $f \in L^{p(\cdot)}(\mathbb{R}^d)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^d)$ holds*

$$\|f\|_{p(\cdot)} \leqslant A\|f^\sharp\|_{p(\cdot)}, \qquad \|f\|_{p'(\cdot)} \leqslant A\|f^\sharp\|_{p'(\cdot)}.$$

The following theorems are important tools in elasticity and fluid mechanics since they enable to estimate the Sobolev norm of the velocity in terms of the symmetric part of the velocity gradient.

Theorem 8.11 (Korn, whole space). *Let p, φ be as is Corollary 8.8. If φ is of class $\mathcal{A}$, then there exists $A > 0$, such that for all $\mathbf{f} \in (W^{1,p(\cdot)}(\mathbb{R}^d))^d$ there holds*

$$\|\nabla \mathbf{f}\|_{L^{p(\cdot)}(\mathbb{R}^d)} \leqslant A\|\mathbf{Df}\|_{L^{p(\cdot)}(\mathbb{R}^d)}.$$

Theorem 8.12 (Korn, bounded domain). *Let p, φ be as is Corollary 8.8 and let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. If φ is of class $\mathcal{A}$, then there exists $A > 0$, such that for all $\mathbf{f} \in (W_0^{1,p(\cdot)}(\Omega))^d$ there holds*

$$\|\nabla \mathbf{f}\|_{L^{p(\cdot)}(\Omega)} \leqslant A\|\mathbf{Df}\|_{L^{p(\cdot)}(\Omega)}. \tag{8.29}$$

We now turn to the examination of Calderón–Zygmund operators on $L^{p(\cdot)}(\mathbb{R}^d)$.

Definition 8.13. A kernel k on $\mathbb{R}^d \times \mathbb{R}^d$ is a locally integrable complex-valued function k, defined off the diagonal. We say that k satisfies standard estimates if there exist $\delta > 0$ and $A > 0$, such that for all distinct $x, y \in \mathbb{R}^d$ and all $z \in \mathbb{R}^d$ with $|x - z| < \frac{1}{2}|x - y|$ holds:

$$\bigl|k(x, y)\bigr| \leqslant A|x - y|^{-d}, \tag{8.30a}$$

$$\bigl|k(x, y) - k(z, y)\bigr| \leqslant A|x - z|^{\delta}|x - y|^{-d-\delta}, \tag{8.30b}$$

$$\bigl|k(y, x) - k(y, z)\bigr| \leqslant A|x - z|^{\delta}|x - y|^{-d-\delta}. \tag{8.30c}$$

In this case we call k a standard kernel.

We say that $T : C_0^\infty(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$, where $\mathcal{D}'$ is the space of distributions, is associated with a kernel k, if for all $f, g \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp}(f) \cap \operatorname{supp}(g) = \emptyset$

$$\langle Tf, g\rangle = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x, y) f(y) g(x)\,\mathrm{d}x\,\mathrm{d}y.$$

If in addition T extends to a bounded, linear operator on $L^2(\mathbb{R}^d)$, then we call T a Calderón–Zygmund operator. By T_ε we denote the operators associated to the truncated kernels k_ε, where $k_\varepsilon(x, y) = 0$ for $|x - y| \leqslant \varepsilon$ and $k_\varepsilon(x, y) = k(x, y)$ for $|x - y| > \varepsilon$. Further let T_* denote the maximal truncated operator, i.e. $(T_* f)(x) = \sup_{\varepsilon>0}(T_\varepsilon f)(x)$.

Theorem 8.14. *Let T be a Calderón–Zygmund operator with kernel k on $\mathbb{R}^d \times \mathbb{R}^d$. Let p, φ be as is Corollary* 8.8. *If φ is of class $\mathcal{A}$, then T and T_* are bounded on $L^{p(\cdot)}(\mathbb{R}^d)$. If in addition k satisfies*

(a) *For every x, $K(x, x - z)$ is integrable over the sphere $|z| = 1$ and its integral is zero.*
(b) *For some $\sigma > 1$ and every x, $|K(x, x - z)|^\sigma$ is integrable over the sphere $|z| = 1$ and its integral is bounded uniformly with respect to x.*

then the operators T_ε are uniformly bounded on $L^{p(\cdot)}(\mathbb{R}^d)$ with respect to $\varepsilon > 0$. Moreover,

$$Tf(x) := \lim_{\varepsilon\to 0^+} T_\varepsilon f(x) = \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^d} k_\varepsilon(x, y) f(y)\,\mathrm{d}y \tag{8.31}$$

exists almost everywhere and $\lim_{\varepsilon\to 0^+} T_\varepsilon f = Tf$ *in $L^{p(\cdot)}(\mathbb{R}^d)$ norm.*

The next theorem is also important for fluid mechanics.

Theorem 8.15. *Let $\Omega \subset \mathbb{R}^d$, $d \geqslant 2$ be a bounded domain with Lipschitz boundary. Let p, φ be as is Corollary* 8.8. *Define*

$$L_0^{p(\cdot)}(\Omega) := \left\{ f \in L^{p(\cdot)}(\Omega)\colon \int_\Omega f(x)\,\mathrm{d}x = 0 \right\}.$$

If φ is of class $\mathcal{A}$, then there exists $A \geqslant 1$ such that for each $f \in L_0^{p(\cdot)}(\Omega)$ there exists $\mathbf{u} \in (W_0^{1,p(\cdot)}(\Omega))^d$ with $\operatorname{div}\mathbf{u} = \mathbf{f}$ *and* $\|\nabla\mathbf{u}\|_{p(\cdot)} \leqslant A\|f\|_{p(\cdot)}$.

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